On the One-Loop Dilatation Operator of Strongly-Twisted $\mathcal{N}$=4 Super Yang-Mills Theory

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von

Friedrich Leonard Zippelius

Präsidentin der Humboldt-Universität zu Berlin

Prof. Dr.-Ing. Dr. Sabine Kunst

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät

Prof. Dr. Elmar Kulke

Gutachter/-innen: 1. Prof. Dr. Matthias Staudacher

2. Dr. Thomas Klose

3. Prof. Dr. Changrim Ahn

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Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018 angegebenen Hilfsmittel angefertigt habe.

Leonard Zippelius

Zusammenfassung

In den letzten beiden Jahrzehnten hat sich $\mathcal{N} = 4$ Super Yang-Mills Theorie (SYM) als vergleichsweise einfache wechselwirkende Quantenfeldtheorie etabliert. Es konnte gezeigt werden, dass $\mathcal{N} = 4$ SYM im sogenannten planaren Limes eine integrable konforme Feldtheorie ist. Diese Erkenntnis wurde im Rahmen der Lösung des Spektralproblems gewonnen, das als die Diagonalisierung des Dilatationsoperators definiert ist. Dieser Operator ist der Teil der konforme Algebra, der Skalentransformationen erzeugt. Seine Eigenwerte sind die anomalen Dimensionen, die unter anderem den Zerfall der Zweipunktkorrelationen charakterisieren.

In jüngerer Zeit wurde vorgeschlagen, dass verwandte Theorien, die man kollektiv als stark getwistete $\mathcal{N} = 4$ SYM bezeichnet, tatsächlich einfacher wären. Wir untersuchen das Spektralproblem dieser Theorien und bestimmen die Eigenwerte des Dilatationsoperators. Dabei ist unsere Analyse auf Einschleifenordnung beschränkt. Dazu leiten wir zunächst den Einschleifendilatationsoperator der stark getwisteten Modelle her und drücken ihn mit Hilfe des Operators der nicht getwisteten Theorie aus. Bemerkenswerterweise ist der Dilatationsoperator nicht diagonalisierbar, da die stark getwisteten Theorien nicht unitär sind. Wir definieren den Begriff des eklektischen Feldinhalts von lokalen zusammengesetzten Operatoren. Eine endliche Potenz des Dilatationsoperators bildet diese lokalen zusammengesetzten Operatoren mit eklektischem Feldinhalt auf null ab. Die Herleitung unterschiedlicher Bethe Ansätze wird präsentiert um die Eigenzustände des Dilatationsoperators zu finden. Des Weiteren schlagen wir eine verkürzte Herleitung vor, die es erlaubt die Bethe Gleichungen aus denen der nicht skalierten Modelle zu gewinnen.

Wir stellen die Lösungen der Bethe Gleichungen vor, wobei wir Sektor für Sektor vorgehen. Im einfachsten Sektor sind wir in der Lage ein Gebiet in der komplexen Ebene abzugrenzen, in der die anomalen Dimensionen liegen müssen. Die dazugehörigen expliziten Lösungen der Bethegleichungen sind leicht zu finden und liegen in dem vorgegebenen Gebiet. Für die größeren Sektoren werden die anomalen Dimensionen der lokalen zusammengesetzten Operatoren berechnet. Dies geschieht systematisch für die Operatoren mit niedriger klassischer Dimension und sporadisch für die mit höherer klassischer Dimension. Für diese Fälle konstruieren wir auch die auftretenden Jordan Blöcke. Des Weiteren diskutieren wir den Einfluss, den die Jordan Blöcke auf die Zweipunktfunktionen der Theorie haben. In einer nicht unitären Theorie ist die Klassifikation der lokal zusammengesetzten Operatoren in Primäroperatoren und Abkömmlinge nicht vollständig und eine dritte Art Operator, nämlich der logarithmische Operator, tritt auf. Die Zweipunktfunktionen des logarithmischen Operators und des dazugehörigen Primäroperators haben nicht die Standardform, sondern enthalten Logarithmen.

Abstract

Over the last two decades, $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) has established a reputation of being the simplest interacting quantum field theory in four dimensions. In the so-called planar limit, $\mathcal{N} = 4$ SYM turned out to be an integrable conformal field theory. Integrability was first found when solving the spectral problem, which is defined as diagonalising the dilatation operator. The latter is the part of the conformal algebra generating scaling transformations. Its eigenvalues are the anomalous dimensions, which i.a. yield the decay of the two-point correlation function.

More recently, it was proposed that a certain non-unitary deformation of $\mathcal{N} = 4$ SYM, the so-called strongly-twisted theories, are actually simpler. We investigate the spectral problem of these theories at one-loop order. We derive the one-loop dilatation operator of the strongly-twisted models and express it in terms of the one of the untwisted theory. Notably, since the strongly-twisted theories are non-unitary, the dilatation operator turns out to be non-diagonalisable. We define the notion of eclectic field content of local composite operators. A finite number of applications of the dilatation operator annihilates these local composite operators with eclectic field content. A derivation of several different Bethe ansätze to find eigenstates of the dilatation operator is presented. Furthermore, we also propose a short-cut to derive the Bethe equations from those of the unscaled models.

We present solutions to the Bethe equations sector by sector. In the simplest sector, we are able to derive a bounded region in the complex plane of the anomalous dimension. The corresponding explicit solutions to the Bethe equations are easy to find. For the larger sectors, the anomalous dimensions of the composite operators are calculated systematically for small classical dimensions. Additionally, we have computed anomalous dimensions for several operators with higher classical dimensions. For these cases, we also present the Jordan blocks of the dilatation operator. Furthermore, we show the impact of Jordan blocks appearing in the dilatation operator on the two-point correlation function of the theory. The classification of local composite operators into primaries and descendants is no longer complete in a non-unitary theory and a third type of operator, named a logarithmic operator, appears. The two-point functions of a logarithmic operator and the corresponding primary are not of the standard form of unitary conformal field theories, but instead contain logarithms.

List of Publications

This thesis is based on the following published paper as well as so far unpublished results.

[1] A. C. Ipsen, M. Staudacher, and L. Zippelius, "The one-loop spectral problem of strongly twisted $\mathcal{N} = 4$ Super Yang-Mills theory," *JHEP* **04** (2019) 044, `arXiv:1812.08794 [hep-th]`.

Many of the equations in this paper are identical to those presented here. All text passages are newly written to present more detail and go into more depth. In addition, several results presented in this thesis (in particular in sections 3.5.4, 4.1, 4.2 and 4.4) are so far unpublished.

Contents

Chapter 1

Introduction

Currently, the best framework to describe elementary particles and their fundamental interactions is local Quantum field theory (QFT). In particular, Yang-Mills theory [2] with the gauge group $SU(3) \times SU(2) \times U(1)$, commonly known as the standard model (SM) of particle physics, accurately and precisely describes the electromagnetic force as well as the weak and strong nuclear forces. Only the fourth fundamental force, gravity, is not incorporated into this theory. The precision of standard model predictions and their agreement with experimental measurements is unprecedented. A prime example is the magnetic moment of the electron, which is known up to a precision of 10^{-12} [3]. In 2012 the Higgs-Boson was found at the Large Hadron Collider (LHC) [4, 5, 6]. The discovery completed the program to discover all particles predicted by the standard model. Thus, there can be no doubt that the standard model is one of the most important and successful theories in physics. Nevertheless, despite this huge success story, the standard model has both conceptual as well as practical shortcomings.

On the conceptual side, the most prominent issue is the lack of a consistent incorporation of gravity. Our best description of gravity is general relativity, which has proved extremely difficult to quantise since it is not (perturbatively) renormalisable. Further issues include the hierarchy problem, dark matter and dark energy, the CP-strong problem, and neutrino oscillations. A large amount of research is invested into finding solutions to these problems. While there are many promising suggestions to alleviate them, one single complete theory still awaits discovery.

On the practical side, explicit and precise calculations of quantities of interest often go beyond the most advanced mathematical tools currently at our disposal. As a consequence, we are usually confined to use approximations, the most prominent of which is perturbation theory. The later uses an expansion in a parameter of the theory, usually taken to be the coupling constant. For this to be meaningful said parameter has to be small. Consequently, strongly coupled theories like quantum chromodynamics (QCD) at low energies, responsible, for example, for quark confinement, cannot be treated within the framework of perturbation theory. Critically, even if the coupling constant is small and perturbation theory should be valid, exact results are often still very limited. Firstly, this is a consequence of increasingly difficult integrals to be computed at each new order in the expansion. Secondly, the number of Feynman diagrams or equivalently integrals that need to be computed also increases in most QFTs with each order.

With the goal of obtaining a better understanding of, and more advanced tools for, calculations in QFTs, several simpler QFTs were investigated. Certainly, one of the most successful of these simpler models is $\mathcal{N} = 4$ super Yang-Mills theory ($\mathcal{N} = 4$ SYM) [7, 8]. $\mathcal{N} = 4$ SYM is also a non-abelian Yang-Mills theory just like the standard model, although

it is based on the gauge group $U(N)$ or $SU(N)$. It has therefore been labelled as a close cousin to the standard model. Furthermore, $\mathcal{N} = 4$ SYM has obtained a reputation as being a simple or even the simplest four-dimensional interacting QFT [9], since many quantities of interest are restricted by symmetry. The theory exhibits both the maximal amount of supersymmetry for a four-dimensional interacting QFT and conformal symmetry. In fact, due to scale invariance, $\mathcal{N} = 4$ SYM is not directly applicable to particle phenomenology and has initially seen only a limited amount of interest.

Interest in $\mathcal{N} = 4$ SYM was renewed and increased significantly with the advent of the Anti-de Sitter/Conformal field theory (AdS/CFT) correspondence [10, 11, 12]. The correspondence proposed by Maldacena in the groundbreaking paper [10] conjectures that a string theory on an AdS background is dual to a CFT on the boundary of the AdS space. Since string theory naturally incorporates gravity, the AdS/CFT duality has opened avenues for progress in quantum gravity through the study of Yang-Mills theory. A further interesting feature of the AdS/CFT duality is that it relates the strongly coupled regimes of the theories to the weakly coupled regimes of the dual theory.

The most prominent example is the conjectured duality of type IIB superstring theory on the AdS background $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the conformal field theory $\mathcal{N} = 4$ SYM. We might, therefore, hope to understand the perturbatively unaccessible strongly coupled regime of, for example, $\mathcal{N} = 4$ SYM by studying weakly coupled string theory. However, as a weak-strong duality, the AdS/CFT duality is also naturally difficult to check, since whenever one side of the duality is accessible to calculations the other is not.

The planar limit, also called the 't Hooft limit, can be taken of $\mathcal{N} = 4$ SYM to simplify the theory [13]. In this limit, we take the rank of the gauge group $SU(N)$ to infinity and the coupling constant g_{YM} to zero, such that the 't Hooft coupling $\lambda = g_{YM}^2 N$ remains fixed. In the planar limit, when computing an amplitude, the only Feynman diagrams that contribute are those that can be drawn on a plane without crossing propagators.

Fascinatingly, $\mathcal{N} = 4$ SYM turns out to be integrable in the planar limit; see [14] for a review. Integrability refers to the existence of sufficiently many conserved charges, which restrict quantities in the theory to such an extent that they can often be computed exactly. For an integrable QFT there are infinitely many conserved charges. Within the context of $\mathcal{N} = 4$ SYM, integrability was first found when solving what is called the spectral problem. This is the problem of diagonalising the dilatation operator, which is the part of the conformal algebra generating scaling transformations. In a unitary CFT, due to conformal symmetry there is a basis of local operators for which the two-point function is particularly simple. For example, for scalar primary operators, the two-point function is just

$$\langle O(x)O(y)\rangle = \frac{C}{(x-y)^{2\Delta}} \tag{1.1}$$

where Δ is the dimension, i.e., the eigenvalue of the dilatation operator corresponding to the operator O. Thus, diagonalising the dilatation operator gives the only theory dependent dynamical information entering the conformal two-point function. However, the spectral problem, in general, is not easily solvable. In 2002, it was observed that the spectral problem of a specific subset of operators in $\mathcal{N} = 4$ SYM to first order in the coupling constant reduces to the problem of diagonalising a spin-chain Hamiltonian [15]. This spin-chain is integrable, leading to the first observation of integrability in $\mathcal{N} = 4$ SYM. As a consequence, computational tools such as the Bethe ansatz [16] can be used to solve the spectral problem. The generalisation of integrability to the complete one-loop dilatation operator of the full theory followed [17, 18, 19]. Although not proven, integrability was conjectured to hold at higher loops as well, leading to an asymptotic Bethe ansatz, which

is valid until the number of loops equals or exceeds the size of the operator [20]. When computing the Feynman diagrams contributing to the two-point function, subtleties in taking the 't Hooft limit arise. Explicitly, once the number of loops becomes too large, non-planar Feynman diagrams are not necessarily suppressed anymore in the 't Hooft limit [21]; see [22, 23, 24] for an explicit example. The contributions of these non-planar diagrams, known as wrapping effects, can be incorporated into the solution of the spectral problem, resulting in the thermodynamic Bethe ansatz (TBA) [25, 26, 27, 28, 29, 30], a review of which can be found in [31]. The state-of-the-art method for computing dimensions is a reformulation of the TBA in what is known as the quantum spectral curve (QSC) [32, 33]; see also [34, 35] for pedagogical introductions. The QSC allowed for the calculation of the anomalous dimensions of some operators up to ten-loops [36], although, in principle, even higher loops are within reach. Interestingly, the QSC also allows yields numerical results at a given finite value of the coupling constant to arbitrary precision [37, 38]. However, beyond one-loop the operators diagonalising the dilatation operator and its explicit form are still not completely known.

Following the discovery of integrability in the spectral problem, it appeared in several other areas of $\mathcal{N} = 4$ SYM. In [39] it was found that scattering amplitudes exhibit Yangian symmetry, an infinite-dimensional symmetry algebra associated with integrability. Wilson loops can also be modified to exhibit Yangian symmetry [40, 41]. Lastly, with some caveats, it was lately also discovered that the action itself is Yangian invariant [42, 43]. Surprisingly, despite all the success of applying integrability to compute quantities in $\mathcal{N} = 4$ SYM, the field-theoretic origin of integrability remain obscure.

One approach to obtain a better understanding of *why* $\mathcal{N} = 4$ SYM is integrable is to look for deformations of the theory that retain integrability. It was suggested to replace the product of fields in the action by a non-commutative, albeit still associative product [44, 45, 46]. This procedure, called the γ-deformation or γ-twist, introduces three real parameters γ_i into the action of $\mathcal{N} = 4$ SYM. The special case where all γ_i are equal is often termed the β-deformation. In fact, the β-deformation was found first and only later generalised to the γ-deformation.

The success of solving the spectral problem in $\mathcal{N} = 4$ SYM was soon extended to its deformations. It was shown how to relate the one-loop dilatation operator of the deformed theories and the undeformed $\mathcal{N} = 4$ SYM. An asymptotic Bethe ansatz was also found to diagonalise the dilatation operator of the twisted theories [47]. Incorporating non-planar diagrams via the TBA is complicated in the deformed theories because specific double-trace couplings have non-vanishing β-functions [48]. Including these double-trace couplings was done in [49, 50, 51] and, in particular, it was realised that the wrapping corrections need to be augmented by what was termed prewrapping corrections. These prewrapping corrections appear already one loop order earlier than the wrapping conditions, justifying the name. Finally, also the QSC was generalised to the deformed models [52].

The introduction of extra parameters, a priori, renders the theory more complicated and thus questions the primary motivation for studying $\mathcal{N} = 4$ SYM in the first place. However, viewing $\mathcal{N} = 4$ SYM as the particular point of the parameter space where all deformation parameters are zero, suggests there might be interesting models at other points of the parameters space. Indeed, it was proposed to allow for complex deformation parameters γ_i and to take the limit $\lambda \to 0$ while letting the twist parameters $q_i = e^{-i\gamma_i/2} \to \infty$, such that $\xi_i = \frac{q_i \lambda}{16\pi^2}$ is kept constant [53, 54]. This procedure was called double-scaling limit and results in new integrable theories with coupling constants ξ_i. Particularly interesting is the fishnet theory, where only one of the new coupling constants is kept non-zero. Another example is the strongly-β-twisted theory that is obtained when all three deformation

parameters are equal.

The double-scaled or strongly-twisted theories possess a number of curious attributes. Most strikingly, the limit requires γ_i to be imaginary, destroying the unitarity of these theories. Furthermore, it was argued that the theories will in general not be conformal anymore [48, 55]. However, it was soon realised that the theories possess conformal fixed points and are almost conformal, in the sense that conformality is preserved up to an exchange of operators consisting of only two elementary fields. On the positive side, the double-scaling procedure yields a specific limited set of interaction terms. In particular, in the fishnet model only a single ϕ^4 vertex with a specific ordering of the fields exists. The form of the interaction term reduces the number of Feynman diagrams dramatically. It has been claimed that for a given quantity of interest at most a single planar Feynman diagram contributes at each loop order [54]. While for the strongly-β-twisted theory there are additional diagrams, the total number is still considerably less than in unscaled $\mathcal{N} = 4$ SYM. Given the tremendous reduction in Feynman diagrams, the hope has arisen to be able to gain insights into integrability, which can potentially be applied to the untwisted theory.

Different facets of the strongly-twisted theories have been investigated. The integrals appearing in the fishnet theory are being studied in their own right, since they contribute in many QFTs. Long before the theory itself was suggested, the integrability of the Feynman diagrams was shown, in the sense that they obey a certain star-triangle relation [56]. Lately, it was conjectured that they could be rewritten in terms of ladder diagrams [57].

Concerning the spectral problem, tremendous progress was made already in [54], where the asymptotic Bethe equations for the model were suggested. Soon after, the full QSC was implemented in the fishnet model as well [58, 59, 35]. However, one should not to jump to the conclusion that the spectral problem is thus solved in these theories. Since the strongly-twisted theories are non-unitary, their dilatation operator is non-hermitian and thus not necessarily diagonalisable. If it is not, the dilatation operator can only be brought into Jordan normal form, but the QSC and in fact already the Bethe ansatz were developed for diagonalisable operators. It remains unclear what these methods yield for non-diagonalisable operators. The question certainly requires attention, and we indeed find the spectral problem to be more intricate than naively expected.

Furthermore, even if the QSC were able to determine the Jordan normal form of the dilatation operator, one of the main goals of studying the strongly-twisted theories is to understand integrability from a field theory perspective. Taking the double-scaling limit of the QSC provides little insight into this subject matter. We thus take the standpoint that we should first investigate the one-loop spectral problem as thoroughly as possible. Only afterwards should we move on to higher loops.

The contents of the thesis are as follows. We start in chapter 2 by reviewing the physics necessary to understand the work this thesis reports on. In particular, we start with an introduction to conformal field theory. Then we discuss the space-time transformations and the conformal algebra as well as the transformation of fields. Subsequently, we briefly show the most important consequences of non-unitarity, which leads to what is known as a logarithmic CFT. This section contains important discussions on this not widely known topic. Afterwards, we introduce the relevant CFTs, starting with untwisted $\mathcal{N} = 4$ SYM and working our way through to the strongly-twisted models. At the end of the review chapter, we then introduce the appearance of spin-chains in the context of the spectral problem.

The main research is reported in chapters 3 and 4. Chapter 3 displays the derivation of methods used to solve the spectral problem. It consists of the derivation of the dilatation

operator in the strongly-twisted theory and several Bethe ansätze applicable to different sectors of the models. It also includes a shortcut to find Bethe equations for some of the larger sectors. Most, but not all, of the discussion in this chapter can be found already in our paper [1]. Here we will give significantly more details concerning the derivations.

In contrast, large parts of chapter 4 are so far unpublished. In this chapter, we analyse the derived Bethe equations and show explicit solutions. We start with the simplest sector. Here we work out some explicit results as well as derive a bound for the anomalous dimensions. Afterwards, we progress to increasingly complex sectors, although if we introduce too many different fields, the spectrum becomes trivial, a phenomenon we called eclecticism. In chapter 4, we discuss mainly non-eclectic sectors. In the last two sections of chapter 4, we return to the impact of non-unitarity and discuss some issues and open questions concerning, in particular, higher-loops. Finally, we conclude in chapter 5.

The twisted, but still unscaled, Bethe equations from [47] have been translated into our conventions and can be found in Appendix A. Appendix B contains a proof of the discovery that operators with eclectic field content have zero anomalous dimension. Related to this phenomenon, Appendix C shows another scenario of zero anomalous dimension. Some further explicit solutions are referred to Appendix D.

Chapter 2

Conformal Field Theories and Integrable Spin Chains

In this chapter, we review previous work relevant to the topics discussed in this PhD-thesis. We start in the first section 2.1 with a discussion of conformal symmetry. We discuss what is understood as a conformal transformation. Furthermore, we introduce the conformal algebra and its unitary representations in which the dynamical variables - the fields - transform in a conformal field theory (CFT). Afterwards, we highlight specific differences between unitary and non-unitary CFT's, the latter of which contain a subclass called logarithmic CFTs (log CFTs). In section 2.2, we introduce the prime example of an integrable interacting quantum field theory in four dimensions, namely planar $\mathcal{N} = 4$ super Yang-Mills theory. Specific deformations also called twists of $\mathcal{N} = 4$ SYM that preserve integrability are introduced as well. At the end of 2.2 we then take a specific limit of the deformed cousins of $\mathcal{N} =$ SYM to arrive at the strongly-twisted theories, we wish to investigate. In section 2.3, we show the relation between correlation functions of local operators from these theories and integrable spin chains. Throughout the chapter, several conventions are introduced.

2.1 Conformal Field Theory

The theories we want to study in this thesis exhibit invariance under a larger symmetry group than under the Poincaré group, namely the conformal group. In this section, we provide a brief review of conformal symmetry and how it helps to compute interesting quantities in a given theory. In the process, we define the dilatation operator and its eigenvalues, the spectrum of dimensions, which are the main objects of study in the later chapters of this thesis. We show how the dimensions enter correlations functions of local operators in a Conformal Field Theory (CFT), which illustrate the importance of the anomalous dimensions. Finally, we finish the section by discussing specific particularities of the theories we study in this thesis, which leads to the notion of logarithmic CFTs.

2.1.1 Conformal Symmetry

As mentioned above the theories of interest in this thesis are covariant under conformal transformations, which we will now discuss. Our treatment is standard and similar discussions can be found for example in the classic text book [60] or the lecture notes [61, 62, 63]. A conformal transformation is defined as a spacetime transformation $x^\mu \to x'^\mu$

under which the metric is left invariant up to a local scale factor

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = \Omega(x) g_{\mu\nu}\,, \tag{2.1}$$

where the first equality is just a reminder of the transformation law of the metric under any spacetime transformation and the second equality is the definition of a conformal transformation. In Minkowski space, the special case $\Omega = 1$ corresponds to the Poincaré transformations. Let us look at an infinitesimal transformation $\delta x = x' - x = \xi$ around Minkowski space with the metric $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(1, -1, ...)$. Plugging our transformation into (2.1) we find to first order

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = f(x)\eta_{\mu\nu}\,, \tag{2.2}$$

where we lowered the index on ξ and $f(x)$ is an a priori unrestricted function, which can be related to ξ by taking the trace of the above equation. By applying a further derivative to (2.2) followed by a permutation of dummy indices, taking a linear combination and a contraction[1], it is straight forward to show that

$$2\partial^2 \xi_\mu = (2-d)\partial_\mu f(x)\,. \tag{2.3}$$

Finally, combining (2.2) and (2.3) we find

$$(2-d)\partial_\mu\partial_\nu f(x) = \eta_{\mu\nu}\partial^2 f(x) \quad \text{and} \quad (1-d)\partial^2 f(x) = 0\,, \tag{2.4}$$

where we took the trace to arrive at the last condition for f(x). In this thesis, we are interested in $d = 4$. From (2.4) we can determine that in more than two dimensions $f(x)$ is at most linear in x and hence

$$\xi_\mu = a_\mu + l_{\mu\nu}x^\nu + c_{\mu\nu\lambda}x^\mu x^\lambda\,. \tag{2.5}$$

From the above equations, several constraints can be determined for $l_{\mu\nu}$ and $c_{\mu\nu\lambda}$, which we will only state in the following. Since Poincaré transformations are a subset of the conformal transformations, we immediately see that a^μ is unconstrained corresponding to translations and $l_{\mu\nu}$ has an antisymmetric part $m_{\mu\nu}$ corresponding to rotations and boosts. In addition, $l_{\mu\nu}$ now has a part proportional to the metric, which corresponds to rescalings:

$$l_{\mu\nu} = \lambda\eta_{\mu\nu} + m_{\mu\nu}\,. \tag{2.6}$$

The coefficient of the quadratic term in (2.5) turns out to have four independent terms, often labelled by

$$b_\mu = \frac{1}{4}c^\rho{}_{\rho\mu}\,. \tag{2.7}$$

The corresponding transformation is called a special conformal transformation and its infinitesimal version is given by

$$\xi^\mu = 2b_\rho x^\rho x^\mu - \eta_{\rho\sigma}b^\mu x^\rho x^\sigma\,. \tag{2.8}$$

The corresponding finite versions of the conformal transformations are well known, but we are not going to need them. Hence we are going to omit them from our discussion. Instead, we will next look at the algebra of the generators corresponding to translations, rotations, boosts, dilations and special conformal transformations.

[1] See chapter 4 of [60] for more details.

2.1.2 The Conformal Algebra

So far, in the last section, we have only described the conformal transformations acting on space-time, *not* how they act on the fields[2] of a given theory. In general the fields will transform as well and we have

$$\Phi(x) \to \Phi'(x') = F(\Phi(x))\,, \tag{2.9}$$

where Φ' is the transformed field Φ is the original field and F is some function defining the transformation of the field. The generator G_a of a symmetry transformation labeled by a is defined[3] as

$$-\omega_a G_a \Phi(x) = \Phi'(x) - \Phi(x)\,, \tag{2.10}$$

where ω_a is a corresponding infinitesimal transformation parameter. Assuming for the moment that the transformation of the fields is trivial

$$\Phi'(x') = \Phi(x)\,, \tag{2.11}$$

we find the generators of the conformal group to be

$$P_\mu = -\partial_\mu \tag{2.12}$$
$$L_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.13}$$
$$K_\mu = (2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) \tag{2.14}$$
$$D = -x^\mu \partial_\mu\,. \tag{2.15}$$

The first two are the generators of translation and Lorentz-transformations respectively. The third one is the generator of special conformal transformations. For the first time, we encounter the main object of this thesis: the fourth generator in the list above, which generates scaling transformations, is called the dilatation operator. These generators obey the following commutation rules

$$[L_{\mu\nu}, L_{\rho\sigma}] = (\eta_{\mu\rho} L_{\nu\sigma} + \eta_{\nu\sigma} L_{\mu\rho}) - \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\sigma} L_{\nu\rho} \tag{2.16}$$
$$[L_{\mu\nu}, P_\rho] = (\eta_{\rho\mu} P_\nu - \eta_{\nu\rho} P_\mu) \tag{2.17}$$
$$[D, P_\mu] = P_\mu \tag{2.18}$$
$$[D, K_\mu] = -K_\mu \tag{2.19}$$
$$[P_\mu, K_\nu] = 2(\eta_{\mu\nu} D - L_{\mu\nu}) \tag{2.20}$$
$$[L_{\mu\nu}, K_\rho] = (\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu)\,, \tag{2.21}$$

with all remaining commutators vanishing. This is the definition of the conformal algebra. In fact, it can be shown that the conformal algebra is isomorphic to $\mathfrak{so}(2,4) \cong \mathfrak{su}(2,2)$.

So far, the discussion has been about the special case, where the transformation of the fields is trivial. In general, the operators of a conformal field theory form representations of the conformal group, in complete analogy to the operators of an ordinary relativistic QFT forming representations of the Poincaré group. Most research has been done on unitary representations, which implies that the representations are fully reducible, i.e., they can be written as the direct sum of irreducible representations. Thus, we will first focus on the

[2] Our definition of the term field is broader than just the elementary degrees of freedom. In particular, a field can be a composite object built from several elementary fields of the theory, e.g., the local composite operator O introduced in (2.57)

[3] There are several different conventions concerning factors of i in the definition of the generators. Our definition here is chosen such that in a unitary CFT the dilatation operator is hermitian.

case where the fields come in irreducible representations of the conformal group. For these irreducible representations, we need to add the generator responsible for the transformation of the field to the space-time generator given in (2.12)-(2.15). We call the action of $L_{\mu\nu}$ on a field at $x = 0$ $S_{\mu\nu}$, i.e.,

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0)\,, \tag{2.22}$$

where S is some irreducible representation of the Lorentz group. It follows from the Baker–Campbell–Hausdorff formula and the commutator of P and L that the action of L at a generic point x is

$$L_{\mu\nu}\Phi(x) = L_{\mu\nu}e^{ix^\rho P_\rho}\Phi(0) = (x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x)\,. \tag{2.23}$$

Similiarly we obtain

$$D\Phi(x) = (x^\mu\partial_\mu + \mathcal{D})\Phi(x)\,, \tag{2.24}$$

$$K_\mu\Phi(x) = (\kappa_\mu - 2x_\mu\mathcal{D} - x^\nu S_{\mu\nu} - 2x_\mu x^\nu\partial_\nu + x^2\partial_\mu)\Phi(x) \tag{2.25}$$

for the action of the dilatation operator and the generator of the special conformal transformation respectively. Here $\mathcal{D}$ and κ are in an irreducible representation of D and K acting on fields at $x = 0$.

Let us assume further that we have an irreducible representation of the Lorentz group, then since $\mathcal{D}$ commutes with $S_{\mu\nu}$ we know that $\mathcal{D} = \Delta I$ has to be proportional to the identity. This means, the field $\Phi(0)$ is an eigenstate of $\mathcal{D}$ with eigenvalue Δ

$$[\mathcal{D}, \Phi(0)] = \Delta\Phi(0)\,. \tag{2.26}$$

Since $\mathcal{D}$ is proportional to the identity, (2.19) implies $\kappa = 0$. It follows after a short calculation that under a finite conformal transformation these fields transforms as

$$\Phi^I(x) \to \Phi'^I(x') = \Omega(x)^{-\Delta}R(\Lambda(x))^I_J\Phi^J(x)\,, \tag{2.27}$$

where $\Omega(x)$ is the change in the metric and $R(\Lambda)$ is the corresponding representation of the Lorentz group element Λ. A field with the above transformation behaviour is called a conformal primary field or just a primary. Acting with P^n on a primary will create other fields, which are called descendants and have dimension $\Delta + n$, as can be seen from (2.18). For unitary theories, it is possible to show that all operators are a possibly infinite linear combination of primaries and descendants. However, as we will see in section 2.1.4, this does not necessarily hold for non-unitary theories.

2.1.3 Correlation Functions

So far, we have discussed the action of the conformal transformations on space-time itself and the fields of a CFT. We will now turn to the consequences of conformal invariance on the observables in a CFT. Our main interest concerns the correlation functions of local operators. In this thesis we will be solely discussing the two-point function

$$\langle O_1(x_1)O_2(x_2)\rangle\,. \tag{2.28}$$

Let us assume further that O_1 and O_2 are scalar primaries defined at the end of the last section. A result from general quantum field theory states that the correlation functions of operators fullfill

$$\langle O_1(x_1')O_2(x_2')\rangle = \langle O_1'(x_1')O_2'(x_2')\rangle\,, \tag{2.29}$$

where a prime denotes a quantity after the application of a symmetry transformation. This equation can be derived using the path integral formalism and using the fact that both the measure and the action are left invariant under a symmetry transformation. In the last section, we expressed $O'(x')$ in terms of $O(x)$ letting us rewrite the right hand side of (2.29)

$$\langle O_1(x_1')O_2(x_2')\rangle = \Omega(x_1)^{\Delta_1}\Omega(x_2)^{\Delta_2}\langle O_1(x_1)O_2(x_2)\rangle\,. \tag{2.30}$$

On the left hand side we only transformed the positions of the operator. Invariance under translation and rotation than implies that the correlator can only be a function of the distance of the two points

$$\langle O_1(x_1)O_2(x_2)\rangle = f(|x_1 - x_2|)\,, \tag{2.31}$$

where f is an undetermined funcion.

We can then use (2.29) for a rescaling to determine the function f.

$$\langle O_1(\lambda x_1)O_2(\lambda x_2)\rangle = \lambda^{-\Delta_1-\Delta_2}\langle O_1(x_1)O_2(x_2)\rangle \tag{2.32}$$

This equation implies that

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,, \tag{2.33}$$

where C_{12} is a constant and Δ_1 and Δ_2 are the dimensions of O_1 and O_2 respectively. Lastly, we have the special conformal transformations. To use (2.29) with the special conformal transformation, we need to determine how $|x_1 - x_2|$ transforms under such a transformation. After a few lines of algebra one sees that under a finite special conformal transformation with transformation parameter b

$$|x_1' - x_2'| = |x_1 - x_2|\sqrt{(1 - 2b\cdot x_1 + b^2x_1^2)(1 - 2b\cdot x_2 + b^2x_2^2)}^{-1}\,. \tag{2.34}$$

Consequently, we see with the help of equation (2.29) that the two-point function vanishes unless the two operators have the same dimension. If several primaries of the same dimension exist, we can always choose a basis in which the two-point function between different operators vanishes and only the two-point function of an operator with itself is non-zero. This means the two-point function of operators is

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{\delta_{1,2}}{|x_1 - x_2|^{2\Delta}}\,, \tag{2.35}$$

where $\Delta = \Delta_1 = \Delta_2$ and we normalised the operators, so the coefficient in front is one. The two-point functions of non-scalar primaries are similarly restricted but contain a factor corresponding to the tensor structure of the operators. Differentiation with respect to x_1 or x_2 provides the correlations functions including descendants. Since the correlations functions of descendants can be obtained from those of the primaries, usually the focus is set on primaries only. However, descendants will often appear in the intermediate steps of calculating correlation functions.

We will also show how to constrain the two-point functions in a conformal theory using the infinitesimal transformations. This is the more obvious path to derive the form of two-point functions in logarithmic CFTs. The two-point function has to be annihilated by

the symmetry generators, so we have

$$(P_{1\mu} + P_{2\mu})\langle O_1(x_1)O_2(x_2)\rangle = (\partial_{1\mu} + \partial_{2\mu})\langle O_1(x_1)O_2(x_2)\rangle = 0 \tag{2.36}$$

$$(M_{1\mu\nu} + M_{2\mu\nu})\langle O_1(x_1)O_2(x_2)\rangle = (x_{1\nu}\partial_{1\mu} - x_{1\mu}\partial_{1\nu} + x_{2\nu}\partial_{2\mu} - x_{2\mu}\partial_{2\nu})\langle O_1(x_1)O_2(x_2)\rangle = 0 \tag{2.37}$$

$$(D_1 + D_2)\langle O_1(x_1)O_2(x_2)\rangle = (x_1 \cdot \partial_1 + \Delta_1 + x_2 \cdot \partial_2 + \Delta_2)\langle O_1(x_1)O_2(x_2)\rangle = 0 \tag{2.38}$$

$$(2(x_1^\mu + x_2^\mu)\Delta + 2x_1^\mu(x_1 \cdot \partial_1) + 2x_2^\mu(x_2 \cdot \partial_2) - x_1^2\partial_1^\mu - x_2^2\partial_2^\mu)\langle O_1(x_1)O_2(x_2)\rangle = 0 \tag{2.39}$$

These are simply the infinitesimal versions of the conditions obeyed by conformal two-point functions. Solving these differential equations will lead to (2.35).

The three- and higher point correlation function are also severely restricted by conformal symmetry. As we will not discuss them in this thesis, we will not give any details here. However, it is important to note that the scaling dimensions of the operators appearing in the two-point function also appear in the higher point functions. In fact, one view of conformal field theories is to see them as just a set of data, the so-called conformal data, which consists of the dimensions of local operators and OPE coefficients. Among other things, this has led to the study of theories that do not possess a Lagrangian. Instead of going into more detail with regards to these well studied conformal theories, in the next section, we will relax one of the most important assumptions we made, namely unitarity.

2.1.4 Logarithmic CFTs

In this section, we drop the assumption of unitarity. Explicitly this means that we do not take D to be a hermitian operator anymore. Earlier, we assumed that $\Phi(0)$ forms an irreducible representation of the Lorentz group. For unitary representations, this is a natural assumption to make, since every reducible representation is fully reducible, sometimes also referred to as decomposable or semi-simple. Let's recall that a representation is reducible if it contains an invariant subspace. It is fully reducible if it can be written as a direct sum of irreducible representations or equivalently it can be brought in block-diagonal form, with each block being an irreducible invariant subspace. Put more loosely, a representation is reducible, but indecomposable, if it contains an invariant subspace, but vectors from 'outside' this subspace get mapped into the subspace under the action of the group. A Jordan-block-type structure appears.

Since this concept is critical, let us look at one of the most famous examples. The integers under addition have the following 2-dimensional representation

$$R(z \in \mathbb{Z}) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} . \tag{2.40}$$

This is clearly a representation of the group, since $R(z_1)R(z_2) = R(z_1 z_2)$. It is reducible, since the first unit vector $(1\ \ 0)^T$ spans an invariant subspace, however, it is indecomposable, since it cannot be brought in block-diagonal form, with a one×one block corresponding to the first unit vector[4], by a change of basis. As mentioned above, this cannot happen for unitary representations; however, for non-unitary representations, this is a possibility. Indeed, the theories we investigate in this thesis will have fields transforming in reducible, but indecomposable representations of the conformal group, hence we will discuss the impact of allowing such representations. In particular, since the theory is not unitary anymore, the operator content does not split anymore into primaries and descendants but

[4]This is a fancy way of saying the matrix can not be diagonalised. In our two by two example, the two statements are identical.

instead contains a third type of operator, which was named logarithmic operator. In general, logarithmic conformal field theories are significantly less studied than non-logarithmic ones. Some early results can be found in [64, 65, 66, 67].

In this thesis, we are particularly interested in the structure of the two-point correlation function, our discussion of which follows [68] closely. Note, however, that [68] only discusses two-dimensional theories. Let us assume we have a non-unitary CFT with a non-hermitian *and* non-diagonalisable dilatation operator, meaning we have two local operators O_1 and O_2 for which

$$DO_1 = \Delta O_1 \quad \text{and} \quad DO_2 = \Delta O_2 + O_1\,, \tag{2.41}$$

where Δ is a complex number, which we will call the generalised dimension. Here, O_1 is a primary operator with a so-called logarithmic partner O_2. They have the same dimension, but transform differently under dilations and special conformal transformations. Just as before, we can use the Ward identities to derive differential equations obeyed by the two-point functions. Translations and rotations imply again that the two-point correlation functions are only functions of the variable $|x_1 - x_2|$. Dilations and special conformal transformations give the following four constraints

$$0 = ((x_1 \cdot \partial_1) + (x_2 \cdot \partial_2) + 2\Delta)\langle O_1(x_1)O_2(x_2)\rangle + \langle O_1(x_1)O_1(x_2)\rangle \tag{2.42}$$

$$0 = ((x_1 \cdot \partial_1) + (x_2 \cdot \partial_2) + 2\Delta)\langle O_2(x_1)O_2(x_2)\rangle + \langle O_1(x_1)O_2(x_2)\rangle + \langle O_2(x_1)O_1(x_2)\rangle \tag{2.43}$$

$$\begin{aligned} 0 = {} & (2(x_1^\mu + x_2^\mu)\Delta + 2x_1^\mu(x_1 \cdot \partial_1) + 2x_2^\mu(x_2 \cdot \partial_2) - x_1^2\partial_1^\mu - x_2^2\partial_2^\mu)\langle O_1(x_1)O_2(x_2)\rangle \\ & + 2x_2^\mu\langle O_1(x_1)O_1(x_2)\rangle \end{aligned} \tag{2.44}$$

$$\begin{aligned} 0 = {} & (2(x_1^\mu + x_2^\mu)\Delta + 2x_1^\mu(x_1 \cdot \partial_1) + 2x_2^\mu(x_2 \cdot \partial_2) - x_1^2\partial_1^\mu - x_2^2\partial_2^\mu)\langle O_2(x_1)O_2(x_2)\rangle \\ & + 2x_1^\mu\langle O_1(x_1)O_2(x_2)\rangle + 2x_2^\mu\langle O_2(x_1)O_1(x_2)\rangle\,, \end{aligned} \tag{2.45}$$

in addition to the usual conditions on the correlation function $\langle O_1(x_1)O_1(x_2)\rangle$. Furthermore, $\langle O_2(x_1)O_1(x_2)\rangle$ and $\langle O_1(x_1)O_2(x_2)\rangle$ fullfill identical differential equations.

For conciseness, we will not show how to derive the appropriate functional form of the two-point function, but just give the result. They are given by

$$\langle O_1(x_1)O_1(x_2)\rangle = 0 \tag{2.46}$$

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{-C}{|x_1 - x_2|^{2\Delta}} \tag{2.47}$$

$$\langle O_2(x_1)O_1(x_2)\rangle = \frac{-C}{|x_1 - x_2|^{2\Delta}} \tag{2.48}$$

$$\langle O_2(x_1)O_2(x_2)\rangle = \frac{B\log(\mu^2(x_1 - x_2)^2)}{|x_1 - x_2|^{2\Delta_0}}\,, \tag{2.49}$$

where μ is an energy scale to render the argument of the logarithm dimensionless.

We will not make significant use of these equations in this thesis. Instead, we will favour the form of two-point functions given in (2.35), which is widespread through the literature. So far, it is not obvious why the logarithmic form given in (2.49) can be circumvented. The critical observation is to note that above we have not conjugated one of the operators in the correlation function. Instead, we considered the operators to be real and hence to fulfil the same transformation laws as their complex conjugates. We will discuss the impact of complex conjugation in section 2.3.2. There it will become clear why the logarithmic form of the two-point function is not needed in our analysis. However, it is sensible to check whether we can actually obtain the logarithmic form of the correlation functions after we have brought the dilatation operator in Jordan normal form. We will do exactly that in 4.5.

2.2 $\mathcal{N} = 4$ SYM and its Deformations

In the last four sections, we laid out the basics of conformal field theory. They will be needed in this thesis, since we want to discuss a logarithmic CFT, in particular, a close cousin of the famous $\mathcal{N} = 4$ SYM. In order to describe the theory of interest and because several of our results carry over from $\mathcal{N} = 4$ SYM, we first give a description of undeformed $\mathcal{N} = 4$ SYM in section 2.2.1. The corresponding research is naturally too vast to be suitably treated in full. Hence, we focus only on the material that is important in the context of this thesis. In section 2.2.2 we will introduce deformation parameters into the theory that are chosen in such a way as to conserve integrability and (arguably) conformal symmetry of $\mathcal{N}$=4 SYM. Finally, in section 2.2.3, we then introduce the theories we want to discuss. These theories referred to as strongly-twisted $\mathcal{N} = 4$ SYM, are obtained by taking certain limits of the deformation parameters and the coupling constant.

2.2.1 $\mathcal{N} = 4$ SYM

The field content of $\mathcal{N} = 4$ is given by a gauge field $A_\mu = A_{\alpha\dot{\alpha}}$, four chiral Weyl fermions $\psi_{a\alpha}$, four antichiral fermions $\bar{\psi}_{a\dot{\alpha}}$ and six real- or equivalently three complex scalars ϕ_i, $\phi_i^\dagger$. In addition, we can construct a covariant derivative D_μ from the gauge field as

$$D_\mu = \partial_\mu - ig_{YM}[A_\mu, \,] \,, \tag{2.50}$$

where we also introduced the Yang-Mills coupling constant g_{YM}. In general, our index conventions are as follows. The greek indices μ, ν are space-time vector indices running from zero to three, the spinor-indices $\alpha = 1, 2$ and $\dot{\alpha} = \dot{1}, \dot{2}$ correspond to the two factors of $\mathfrak{su}(2)$, which are isomorphic to the Lorentz algebra[5] and the latin indices $i = 1, 2, 3$ and $a = 1, 2, 3, 4$ are internal flavour indices.

All the fields are matrix-valued and transform in the adjoint representation of a gauge group, which we will take to be $SU(N)$. We leave the matrix structure of the fields implicit, but should keep in mind that strictly speaking $\Phi(x) = \Phi(x)^{\mathfrak{m}} T_{\mathfrak{m}}$, where $T_{\mathfrak{m}}$ are the generators of the gauge group, which fullfill

$$\mathrm{tr}(T_{\mathfrak{m}}) = 0 \qquad T_{\mathfrak{m}}^\dagger = -T_{\mathfrak{m}} \qquad \mathrm{tr}(T_{\mathfrak{m}} T_{\mathfrak{n}}) = \delta_{\mathfrak{mn}} \,. \tag{2.51}$$

Given a gauge transformation $U \in SU(N)$, the elementary fields, including the covariant derivative but *not* the gauge field A_μ, transform as

$$\Phi(x) \to U \Phi(x) U^\dagger \,. \tag{2.52}$$

Taking the commutator of two covariant derivatives we can create a field strength tensor

$$F_{\mu\nu} = \frac{i}{g_{YM}} [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{YM}[A_\mu, A_\nu] \,. \tag{2.53}$$

Given the definition of the field content, we can now present the action of $\mathcal{N} = 4$. There are several different ways to write this action, we will choose the one best suited to

[5] We can replace a space-time index μ with two spinor-indices by multiplication with a $\sigma^\mu_{\alpha\dot{\alpha}} = \{1, \sigma_1, \sigma_2, \sigma_3\}$, where σ_i are the Pauli-matrices.

introduce deformations. The action is

$$\begin{aligned} S = \int d^4x \,\mathrm{tr}\Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D_\mu\phi_i^\dagger)(D_\mu\phi_i) + i\bar{\psi}^{A\dot{\alpha}}(D^\alpha_{\dot{\alpha}}\psi_{A\alpha}) \\ & + \frac{g_{YM}}{\sqrt{2}}\Big(i\epsilon^{ijk}\psi_i^\alpha[\phi_j,\psi_{\alpha k}] + 2i\bar{\psi}^i_{\dot{\alpha}}[\phi_i,\bar{\psi}^{4\dot{\alpha}}] + \text{hermitian conjugate}\Big) \\ & + \frac{g_{YM}^2}{4}\Big(2[\phi^{i\dagger},\phi^{j\dagger}][\phi_i,\phi_j] - [\phi^{i\dagger},\phi_i][\phi^{j\dagger},\phi_j]\Big)\Big\}. \end{aligned} \tag{2.54}$$

Here ϵ^{ijk} is the Levi-Civita symbol. We see the standard kinetic terms augmented by the interactions, which are Yukawa interactions between the fermions and scalars as well as ϕ^4 interactions. The action has two free parameters the gauge coupling g_{YM} and the rank of the gauge group N. It also has an internal $SU(4) \cong SO(6)$ symmetry under which the scalars, fermions and anti-fermions transform in the **6**, **4** and $\bar{\mathbf{4}}$ respectively. Furthermore, the action is invariant under conformal transformations and exhibits maximal $\mathcal{N} = 4$ supersymmetry, hence the name of the theory. The large amount of supersymmetry has significantly contributed to the understanding of the theory. We will, however, not discuss it further, since we are going to break it at least partially when moving to the strongly-twisted theories.

In general, the conformal symmetry of the action does not imply conformal covariance of the full quantum theory. In the path-integral formalism, the functional measure might not be invariant under the corresponding transformations. In fact, conformal symmetry is usually broken by quantum corrections, i.e., it is anomalous. An alternative way to see the anomaly is to notice that quantum field theories are often divergent. This forces the introduction of a scale μ to regularise the theory, which naturally can not be scale-invariant. In general, the parameters of the theory then develop a scale dependence, which results in the observables being scale-dependent as well. For a fixed rank N of the gauge group the theory has only one free parameter, the gauge coupling g_{YM}. Its scale dependence is given by the β-function

$$\beta = \mu\frac{dg_{YM}}{d\mu}\,. \tag{2.55}$$

For $\mathcal{N}$=4 SYM, the β-function is widely believed to vanish, both perturbatively to all loops as well as non-perturbatively [69, 70, 71, 72, 73, 74]. Thus, the conformal symmetry of the theory continues to hold in the full quantum theory.

In many instances, it is useful to combine the two parameters mentioned in the last paragraph into the so-called t'Hooft coupling constant

$$\lambda = g_{YM}^2 N\,. \tag{2.56}$$

We can then expand quantities in λ and N^{-1}. Of particular interest to us is the large N or planar limit, in which we take N to infinity and g_{YM} to zero such that λ is fixed [13]. In this limit, when computing an amplitude, only planar Feynman diagrams contribute, i.e. diagrams which can be drawn on the plane without propagators crossing, since all other diagrams come with additional inverse powers of N. This effect can be seen from the double line notation, where the colour flow is encoded in the propagators. In this notation, each propagator is drawn with two lines, which simplifies counting powers of N, since all closed single lines just contribute a factor of $\delta^i_i = N$. For details on the double line notation see [13]. In the planar limit, $\mathcal{N} = 4$ SYM is integrable, which allows for the computation of anomalous dimensions and OPE-coefficients. Starting from section 2.3, we will always consider the planar limit unless explicitly specified otherwise.

The main observables we will study in this thesis are two-point functions of local gauge-invariant single-trace operators. These operators are

$$O(x) = \mathrm{tr}\left(D^{k_1}\Phi_1(x)D^{k_2}\Phi_2(x)\cdots D^{k_L}\Phi_L(x)\right)\,, \tag{2.57}$$

where each Φ is taken from the set

$$\Phi \in \{F^{\mu\nu}, \phi_i, \phi_i^\dagger, \psi_{a\alpha}, \bar{\psi}_{a\dot{\alpha}}\}\,. \tag{2.58}$$

Single-trace refers to the exclusion of products of traces. The sum of two single-trace operators is still a single-trace operator.

It is useful to express the elementary fields of the theory in terms of oscillators [75, 76]. Let us introduce the following set of raising operators

$$\{a_1^\dagger, a_2^\dagger, b_1^\dagger, b_2^\dagger, c_1^\dagger, c_2^\dagger, c_3^\dagger, c_4^\dagger\} \tag{2.59}$$

and the corresponding lowering operators. We take these operators to fullfill the commutation relations

$$[a^i, a_j^\dagger] = \delta_j^i \quad [b^i, b_j^\dagger] = \delta_j^i \quad \{c^i, c_j^\dagger\} = \delta_j^i\,. \tag{2.60}$$

Finally we also define a vacuum state $|0\rangle$ that is annihilated by all lowering operators and create a vector space by acting with the raising operators. We can than define a map from the elementary field (2.58) to states in this vector space as follows.

$$\begin{aligned}
&\phi_i \to c_i^\dagger c_4^\dagger\,|0\rangle \\
&\phi_i^\dagger \to c_j^\dagger c_k^\dagger\,|0\rangle\,,\ \text{where } j,k\neq i \text{ and } j,k\neq 4 \\
&\psi_{i\alpha} \to a_\alpha^\dagger c_i^\dagger\,|0\rangle \\
&\bar{\psi}_{i\dot{\alpha}} \to b_{\dot{\alpha}}^\dagger c_j^\dagger c_k^\dagger c_l^\dagger\,|0\rangle\,,\ \text{where } j,k,l\neq i \\
&\partial_{\alpha\dot{\alpha}} \to a_\alpha^\dagger b_{\dot{\alpha}}^\dagger\,,
\end{aligned} \tag{2.61}$$

with additional expressions for the field strength tensor, which we will not need. There is a deep connection to the symmetry group of $\mathcal{N}=4$ SYM and the above oscillators, in fact, the above vector space is a representation of the symmetry algebra. However, since we will break the symmetry, we will not discuss it here and instead refer to the explanations in [75, 76]. Nevertheless, it is important to keep the above map from the elementary fields to the vector space in mind, since a natural expression of the dilatation operator is given in terms of its action on the raising operators. We give the dilatation operator of $\mathcal{N}=4$ SYM at the beginning of chapter 3. This ends our discussion of $\mathcal{N}=4$ SYM, and we will next turn to deforming this theory such that (conformality and) integrability remain intact.

2.2.2 Twisting

In this section, we describe how to introduce three parameters γ_1, γ_2 and γ_3 into $\mathcal{N}=4$ SYM, such that conformal symmetry and integrability of the theory remain intact [44, 46]. Our discussion follows [48, 49, 50]. The twist parameters occur in the action in a very specific way based on the following star-product

$$A * B = ABe^{\frac{i}{2}(\mathbf{q}_A\wedge\mathbf{q}_B)}\,. \tag{2.62}$$

The vectors $\mathbf{q}_A$ and $\mathbf{q}_B$ are the Cartan charges under the SU(4) R-symmetry of the fields A and B respectively. They can be read off from table 2.1. The wedge product is defined as

$$\mathbf{q}_A\wedge\mathbf{q}_B = \mathbf{q}_A^T C\mathbf{q}_B \tag{2.63}$$

via a twist matrix

$$C = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{2.64}$$

For now, we assume that the γ_i are real parameters, implying that the star-product only

A	ψ_1	ψ_2	ψ_3	ψ_4	$\mathcal{F},\overline{\mathcal{F}}$	ϕ_1	ϕ_2	ϕ_3
$\mathbf{q}^1_A$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	1	0	0
$\mathbf{q}^2_A$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	1	0
$\mathbf{q}^3_A$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	1

Table 2.1: The charge vectors of the different fields in γ-twisted $\mathcal{N}$=4 SYM. Conjugate fields have the opposite charges. Gauge fields do not carry and derivatives do not add charge.

differs from the normal product by a phase. The star-product can be generalised to more than two fields by imposing associativity. The charge vectors $\mathbf{q}$ of a product of fields is simply the sum of the individual charge vectors. For three fields we have

$$A * B * C = ABC e^{\frac{i}{2}(\mathbf{q}_A \wedge \mathbf{q}_B + \mathbf{q}_A \wedge \mathbf{q}_C + \mathbf{q}_B \wedge \mathbf{q}_C)}. \tag{2.65}$$

Having introduced the star-product, we can write down the action. It is obtained from the action of $\mathcal{N}$=4 by replacing the products in some of the (anti-)commutators by star products. To be explicit a starred commutator is simply

$$[A, B]_* = A * B - B * A \tag{2.66}$$

and similarly for anticommutators. The action of γ-deformed $\mathcal{N}=4$ SYM is

$$\begin{aligned} S = \int d^4x \,\mathrm{tr}\Big\{ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D_\mu \phi_i^\dagger)(D_\mu \phi_i) + i\bar{\psi}^{A\dot{\alpha}}(D^\alpha_{\dot{\alpha}}\psi_{A\alpha}) \\ & + \frac{g_{YM}}{\sqrt{2}}\Big(i\epsilon^{ijk}\psi_i^\alpha[\phi_j, \psi_{\alpha k}]_* + 2i\bar{\psi}^i_{\dot{\alpha}}[\phi_i, \bar{\psi}^{4\dot{\alpha}}]_* + \text{hermitian conjugate}\Big) \\ & + \frac{g_{YM}^2}{4}\Big(2[\phi^{i\dagger}, \phi^{j\dagger}]_*[\phi_i, \phi_j]_* - [\phi^{i\dagger}, \phi_i][\phi^{j\dagger}, \phi_j]\Big)\Big\}. \end{aligned} \tag{2.67}$$

The special case of the γ-deformation, where we set all γ_i equal, is called β-deformation. The introduction of the twist parameters in the action in general breaks the supersymmetry. Furthermore, the SU(4) R-symmetry is reduced to a $U(1)^3$ times some remnant discrete symmetry, which we will discuss later. However, in the case of the β-deformation, one supersymmetry survives, and the β-deformation is still $\mathcal{N}=1$ supersymmetric. Common combinations of the twist parameters appearing in the interaction part of the action are

$$\gamma_1^\pm = \pm\frac{1}{2}(\gamma_2 \pm \gamma_3) \qquad \gamma_2^\pm = \pm\frac{1}{2}(\gamma_3 \pm \gamma_1) \qquad \gamma_3^\pm = \pm\frac{1}{2}(\gamma_1 \pm \gamma_2). \tag{2.68}$$

The twist parameters appear in the interaction terms of the action. As such, when computing Feynman diagrams, each vertex a priori comes with a twist-factor and one might fear that calculations become very complicated. However, calculations are significantly simplified, due to a theorem by Filk [77]. This theorem states that the twist factors can

be consistently pulled out of the inside of a planar diagram of elementary excitations and only depend on the external legs. However, a little more care needs to be taken, if the diagram contains local composite operators, like the ones we are going to look at. Filk's theorem can be applied, only if after removing the composite operator from the diagram we are left with a single trace and not a double trace diagram. We obtain a double-trace diagram if we can in double-line notation cut the diagram into two disjoint pieces, by inserting a single double-trace vertex. See [49] for a more detailed explanation. At one-loop order, which is what we are going to focus on in this thesis, this can easily be avoided by only considering operators of length three or higher. The one-loop diagrams relevant for calculating the dilatation operator for those operators are then just the diagrams in $\mathcal{N}=4$ SYM multiplied by the additional factor $\Phi(A_1 * \cdots * A_L)\Phi(B_1 * \cdots * B_L)$, where A_i and B_i are the elementary fields of the composite operator and Φ is the phase factor from the star-product.

In the calculation of loop diagrams for the γ-deformation UV divergencies occur. These divergencies have to be cancelled by counter-terms, which leads to running coupling constants. In particular, in the γ-deformations, one has to consider certain double-trace couplings and finds that for generic values of γ the β-functions for these couplings do not vanish. This implies that the γ-deformation is not conformal at any real value of the double-trace coupling constants. Fixed points of the β-functions exist in the complex plane [55], however, complex coupling constants are usually not considered. Nevertheless, if the coupling constants are taken to lie at the fixed points and hence the double trace vertices have been included in the theory the full quantum theory is conformal. Since the double-trace terms are suppressed at one-loop for operators with at least three fundamental fields, we can safely ignore them when restricting to operators of length three or higher.

2.2.3 Scaling

In this section, we will finally introduce the theories we are going to investigate in this thesis. The suggestion in [53] was to start with γ-deformed $\mathcal{N}$=4 SYM and take the limits

$$\lambda \to 0 \qquad \text{and} \qquad q_j = e^{-\frac{i}{2}\gamma_j} \to \infty\,, \tag{2.69}$$

while keeping the product $\xi_i = \lambda q_i$ fixed[6]. The limit of the twist factor implies that now the twist parameters γ_i become complex. In turn, the star-product differs from the usual product by more than a phase and the Lagrangian is no longer real. Thus, we are now working with a non-unitary theory. There are different theories described in the following resulting from the double-scaling limit defined in equation (2.69).

The general strongly-twisted γ-deformation arises in the limit with all ξ_i taking generic values. The gauge-field and the gluino ψ_4 decouple, but the other fields interact via Yukawa and ϕ^4 interactions. The Lagrangian of strongly-γ-twisted $\mathcal{N}=4$ is

$$\begin{aligned}\mathcal{L} = N_C\,\mathrm{tr}\Big(&-\frac{1}{2}\partial^\mu\phi_i^\dagger\partial_\mu\phi_i + \bar\psi_{A\dot\alpha}\partial^{\alpha\dot\alpha}\psi_\alpha^A + \xi_1^2\phi_2^\dagger\phi_3^\dagger\phi_2\phi_3 + \xi_2^2\phi_3^\dagger\phi_1^\dagger\phi_3\phi_1 + \xi_3^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2 \\ &+ i\sqrt{\xi_1\xi_2}(\psi_2^\alpha\phi_3\psi_{1\alpha} + \bar\psi_2^{\dot\alpha}\phi_3^\dagger\bar\psi_{1\dot\alpha}) + i\sqrt{\xi_2\xi_3}(\psi_3^\alpha\phi_1\psi_{2\alpha} + \bar\psi_3^{\dot\alpha}\phi_1^\dagger\bar\psi_{2\dot\alpha}) \\ &+ i\sqrt{\xi_1\xi_3}(\psi_1^\alpha\phi_2\psi_{3\alpha} + \bar\psi_1^{\dot\alpha}\phi_2^\dagger\bar\psi_{3\dot\alpha})\Big).\end{aligned} \tag{2.70}$$

The simplest strongly-twisted theory has been named fishnet theory. It is obtained from (2.70), by setting two of the $\xi_i = 0$, while taking $\xi_3 = \xi$ as the new coupling. The

[6] Up to two of the ξ_i are allowed to go to zero.

fishnet is simply a ϕ^4 theory and its Lagrangian is given by

$$\mathcal{L} = \frac{N_C}{2} \operatorname{tr}(-\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2) \,. \tag{2.71}$$

Most of the fields have decoupled in this limit. The gauge field is not charged under the Cartan-charges of the R-symmetry and can thus not pick up factors of q. Since we also sent the gauge coupling to zero, the gauge field decouples. In addition, in this limit, all fermions and the third scalar decouple as well, and we are left with only two-scalars, which interact via a chiral ϕ^4 interaction. The reason this theory has become a subject of interest is that the chirality of the vertex implies that the number of planar diagrams one can draw is very limited. They are all of the form of a square lattice, hence the name fishnet theory.

A third strongly-twisted theory can be obtained from (2.70) by setting all coupling constants equal: $\xi_1 = \xi_2 = \xi_3 = \xi$. This theory is equally well obtained through the double scaling procedure of the β-deformed theory and is hence called strongly-β-twisted. Its action is

$$\begin{aligned}\mathcal{L} = N_C \operatorname{tr}\Big(&- \frac{1}{2}\partial^\mu \phi_i^\dagger \partial_\mu \phi_i + \bar{\psi}_{A\dot{\alpha}} \partial^{\alpha\dot{\alpha}} \psi_\alpha^A + \xi^2 \phi_2^\dagger \phi_3^\dagger \phi_2 \phi_3 + \xi^2 \phi_3^\dagger \phi_1^\dagger \phi_3 \phi_1 + \xi^2 \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 \\ &+ i\xi(\psi_2^\alpha \phi_3 \psi_{1\alpha} + \bar{\psi}_2^{\dot{\alpha}} \phi_3^\dagger \bar{\psi}_{1\dot{\alpha}} + \psi_3^\alpha \phi_1 \psi_{2\alpha} + \bar{\psi}_3^{\dot{\alpha}} \phi_1^\dagger \bar{\psi}_{2\dot{\alpha}} + \psi_1^\alpha \phi_2 \psi_{3\alpha} + \bar{\psi}_1^{\dot{\alpha}} \phi_2^\dagger \bar{\psi}_{3\dot{\alpha}})\Big).\end{aligned} \tag{2.72}$$

We can immediately see that the hermitian conjugates of the interaction terms in all three Lagrangians (2.70), (2.71) and (2.72) are absent. As a consequence, the Lagrangian is not real, and the theory is not unitary. A few conceptual difficulties arise, in particular, the non-hermitian dilatation operator is not necessarily diagonalisable. Put differently, we might be dealing with a logarithmic conformal field theory, as described in section 2.1.4. As we will show, this is indeed the case.

For simplicity, we will mainly focus on the fishnet theory and the strongly-β-twisted theory in this thesis. The results we obtain are, however, rather easily generalised from the β-twisted to the γ-twisted theory. As mentioned earlier, the $SU(4)$ flavour symmetry of $\mathcal{N} = 4$ SYM is broken to $U(1)^3$ symmetry in the deformed theories and a $U(1)^2$ symmetry in the case of the fishnet theory. These $U(1)$ symmetries are the global symmetries of the ϕ_i, under which

$$\phi_i \to e^{i\theta} \phi_i \,. \tag{2.73}$$

However, there is an additional discrete remnant of the $SU(4)$ symmetry as well. For the β-twisted theory the discrete symmetry group is generated by the following three transformations

$$\phi_1 \to \phi_2 \,, \quad \phi_2 \to \phi_3 \,, \quad \phi_3 \to \phi_1 \tag{2.74}$$

$$\phi_1 \to \phi_1^\dagger \,, \quad \phi_2 \to \phi_2^\dagger \,, \quad \phi_3 \to \phi_3^\dagger \tag{2.75}$$

$$\phi_1 \to \phi_1^T \,, \quad \phi_2 \to \phi_3^T \,, \quad \phi_3 \to \phi_2^T \,, \tag{2.76}$$

and the fermions transform accordingly. Here the superscript T denotes the transpose of the matrix. Creating the multiplication table, or equivalently just by inspection, we see that the first and the third of these create an S_3 symmetry, while the second one generates an independent S_2. Thus, the full internal symmetry group of the β-twisted theory is

$$G = (S_3 \times S_2) \ltimes U(1)^3 \,. \tag{2.77}$$

In particular, the internal symmetry group is a semi-direct product, since the $U(1)$ transformations are not independent of the permutations of the $S_3 \times S_2$.

For the fishnet theory, all group elements of the discrete symmetry group can be obtained from products of

$$\phi_1 \to \phi_2^T\,, \quad \phi_2 \to \phi_1^T \tag{2.78}$$

$$\phi_1 \to \phi_2^\dagger\,, \quad \phi_2 \to \phi_1\,. \tag{2.79}$$

Together, these two group elements generate the dihedral group D_4,i.e., the symmetries of a square. The first one corresponds to reflections and the second one to rotations. Again the $U(1)^2$ factors are not independent, and the full internal symmetry group is given by a semidirect product

$$G = D_4 \ltimes U(1)^2\,. \tag{2.80}$$

2.3 Spin Chains and Two-Point Functions

We will now turn to spin-chains and their relation to correlation functions of conformal field theories. In particular, we will see that a large part of the computation of two-point functions of $\mathcal{N}=4$ SYM and its deformations can be mapped to the problem of diagonalising the Hamiltonian of an integrable spin-chain.

2.3.1 Spin chains

A spin-chain is a one-dimensional lattice of smaller components called the spin-chain sites. Each spin-chain site is associated with a vector space, and the Hilbert space of the spin-chain is the tensor product of said vector spaces. To illustrate the concept, let us use the most prominent example, namely the XXX-Heisenberg spin-chain. In this example, each spin-chain site[7] is just $\mathbb{C}^2$. The whole spin-chain is then just the L-fold tensor-product of the spin-chain sites, where L is the number of sites also referred to as the length of the spin-chain. For the typical example of the Heisenberg spin-chain we have

$$V = \left(\mathbb{C}^2\right)^{\otimes L}\,, \tag{2.81}$$

with V being the Hilbert-space of the Heisenberg spin-chain. An excellent introduction into tensor products in the context of the Heisenberg spin-chain can be found in [78].

In addition to the Hilbert-space we are of course also interested in observables of our model, i.e., operators acting on the states of our spin-chain, most notably the Hamiltonian. Unless specified otherwise the Hamiltonians we investigate in this thesis have the nice property that they only contain nearest neighbour interactions. Explicitly, the Hamiltonian H can be written in terms of a Hamiltonian density $\mathcal{H}$ as

$$H = \sum_{j=1}^{L} I_1 \otimes \ldots \otimes I_{j-1} \otimes \mathcal{H}_{j,j+1} \otimes I_{j+2} \otimes \ldots \otimes I_L\,, \tag{2.82}$$

where I_k is the identity acting on the kth spin-chain site and $\mathcal{H}_{j,j+1}$ acts only on the tensor product of the spin-chain sites j and $j+1$, while leaving the rest of the spin-chain invariant. Here, as always, we assume periodic boundary conditions: $L+1=1$. Our example of the Heisenberg spin-chain also exhibits this nearest neighbour interaction and the Hamiltonian density can be written as

$$\mathcal{H}_{l,l+1} = 2(I_{l,l+1} - P_{l,l+1})\,, \tag{2.83}$$

[7] We will use the name spin-chain site for both the point in the one-dimensional lattice as well as the vector space associated to it.

where I is again the identity and P is the permutation operator exchanging the states of the lth and $(l+1)$th spin-chain sites.

The above Hamiltonian density (2.83) can also be written using the Pauli matrices and identity. In fact, that is the form in which it is usually introduced see for example [79] or again [78]. Note that the Pauli matrices acting on $\mathbb{C}^2$ form a representation of $\mathfrak{su}(2)$. In fact, this is more general, and usually, the individual spin-chain sites form a representation under some symmetry group. While symmetries play a fundamental role in integrable models and spin-chains, the core part of this thesis can be understood just as well without any reference to the occurring symmetries and representations. For this reason, we will simply allow for general vector spaces at each spin-chain site.

2.3.2 Two-Point Functions

The goal of this thesis is to describe the one-loop dilatation operator of the strongly-twisted theories and in particular, to find its eigenvalues: the anomalous dimensions. These dimensions most readily appear in the two-point functions, as we discussed in section 2.1.3. Let us remind ourselves of the form of two-point functions of scalar operators in a CFT

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{\delta_{1,2}}{|x_1-x_2|^{2\Delta}}\,. \tag{2.84}$$

In general Δ will be a function of the coupling constant ξ and we can expand it as $\Delta = \Delta_0 + \gamma(\xi)$. Here, Δ_0 is the classical dimension; it is just the sum of the dimensions of the individual fields in an operator. The anomalous dimension, γ, is a function of the coupling. In this thesis, we are interested in the one-loop contribution to γ. If we expand the two-point function for small coupling ξ we obtain

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{\delta_{1,2}}{|x_1-x_2|^{2\Delta_0}}(1-\gamma\ln(\Lambda^2|x_1-x_2|^2))\,, \tag{2.85}$$

where Λ is some UV scale to render the argument of the logarithm dimensionless. Λ will have to be introduced to regulate the UV divergences of the theory and should drop out of the full non-perturbative observables of the theory.

The expansion of the two point function (2.85) provides an avenue to calculate the eigenvalues of the dilatation operator, since for primaries

$$[D, O(0)] = \Delta O(0) = (\Delta_0+\gamma)O(0)\,. \tag{2.86}$$

Let us therefore calculate the two point function of two local-composite operators consisting only of ϕ_1 and ϕ_2 in the fishnet theory, i.e.

$$\langle \mathrm{tr}(\phi^{\dagger J_1}(x_1)\cdots\phi^{\dagger J_L}(x_1))\,\mathrm{tr}(\phi_{I_1}(x_2)\cdots\phi_{I_L}(x_2))\rangle \tag{2.87}$$

In order to calculate the two-point functions we use Feynman diagrams. The fishnet theory only has one chiral four-valent vertex as can be seen from (2.71). We need to insert the vertex into the correlator and Wick contract the fields in a planar manner, which yields

$$\begin{aligned}&\langle \mathrm{tr}(\phi^{\dagger J_1}(x_1)\cdots\phi^{\dagger J_L}(x_1))\,\mathrm{tr}(\phi_{I_1}(x_2)\cdots\phi_{I_L}(x_2))\rangle_{\mathrm{one-loop}}\\ &= 2i\xi^2\frac{\delta^{J_1}_{I_1}\cdots\delta^{J_{k-1}}_{I_{k-1}}\delta^{J_{k+2}}_{I_{k+2}}\cdots\delta^{J_L}_{I_L}}{(4\pi^2)^{L+2}(x-y)^{2(L-2)}}\delta^2_{I_k}\delta^1_{I_{k+1}}\delta^{J_k}_1\delta^{J_{k+1}}_2\int\frac{d^4z}{(z-x_1)^4(z-x_2)^4} + \text{ cyclic permutations},\end{aligned} \tag{2.88}$$

where the cyclic permutations indicate that we have to cyclically permute the indices of one of the operators to find all contractions. The Kronecker-δ impose conditions on the flavour structure of the two operators.

Next, we have to evaluate the integral. As it stands, the integral is UV-divergent around the two points x and y and this divergence has to be regulated. Most calculations are done in dimensional regularisation, but in this particular case, it is easier to introduce a UV-cutoff Λ. The dominant part of the integral then comes from the areas around the two points x and y, where we can approximate the integral as

$$i\int \frac{d^4z_E}{(z_E-x)^4(z_E-y)^4} = \frac{2i\pi^2}{(x-y)^4}\int_{\Lambda^{-1}}^{|x-y|}\frac{dr}{r} = \frac{2i\pi^2}{(x-y)^4}\ln(\Lambda^2(x-y)^2) + \text{finite}\,, \quad (2.89)$$

where we already did a Wick rotation and indicated that there might be additional terms which are finite in the $\Lambda\to\infty$ limit. These additional terms could also include terms for choosing the limits differently. Critically, these finite terms are independent of x and y and can be absorbed by choosing the counterterms in the renormalisation procedure appropriately. The factor of two comes from the fact that we have two regions around x and y, which diverge, while the factor of π^2 finds its origin in the angular part of the integration.

We redefine our local operators to absorb a factor of $(4\pi^2)^{-L}$ and obtain for the rescaled operators

$$\begin{aligned}\langle \bar{O}^{J_1\cdots J_L}(x_1)O_{I_1\cdots I_L}(x_2)\rangle_{\text{one-loop}} &= \frac{-\xi^2\ln(\Lambda^2(x-y)^2)}{8\pi^2(x_1-x_2)^{2L}}\\ \times(\delta^{J_1}_{I_1}\cdots\delta^{J_{k-1}}_{I_{k-1}}\delta^{J_{k+2}}_{I_{k+2}}\cdots\delta^{J_L}_{I_L}\delta^2_{I_k}\delta^1_{I_{k+1}}\delta^{J_k}_1\delta^{J_{k+1}}_2 &+ \text{cyclic}),\end{aligned} \quad (2.90)$$

where we only show the one-loop contributions. It is a convention to absorb a factor of $1/16\pi^2$ into the coupling constant, and we will do so as well. Concretely, from now on we will simply refer to $\xi^2/16\pi^2$ as ξ^2. We observe that the two-point functions of operators with definite flavour content are no longer of the form suitable for primaries once the one-loop contribution is taken into account. This phenomenon is usually called operator mixing. The primaries are then given by the linear combinations, which diagonalise the product of Kronecker-δ and the anomalous dimensions are the corresponding eigenvalues.

Since the space-time part is always the same, it is easiest to separate the problem of diagonalising the mixing matrix from that of doing the Feynman integrals. The mixing matrix is first determined by the Feynman integral calculation above and then diagonalised by mapping the system to a spin-chain. The basis states of the single spin-chain sites are just the different elementary fields with derivatives acting on them. The mixing matrix has invariant subspaces, also called closed sectors [18], like the two-scalar sector for which we calculated the Feynman diagrams above, where the two basis states are ϕ_1 and ϕ_2 and the spin-chain sites are just a $\mathbb{C}^2$. The spin-chain is constructed from the spin chain sites as described in section 2.3.1.

A critical observation is that the mapping from single-trace operators imposes constraints on the corresponding states of the spin-chain. We clearly have to impose periodic boundary conditions to capture the cyclic structure of the trace. However, we also have to impose invariance under cyclic shifts, i.e., invariance under

$$\mathcal{V}_1\otimes\mathcal{V}_2\otimes\cdots\otimes\mathcal{V}_L\to\mathcal{V}_L\otimes\mathcal{V}_1\otimes\cdots\otimes\mathcal{V}_{L-1}\,, \quad (2.91)$$

where we indicated the vector space of spin-chain site i by $\mathcal{V}_i$. Often, the eigenstates of the spin-chain are found without this condition imposed, and all non-invariant states are

projected out by imposing a so-called zero-momentum condition afterwards, see chapter 3 for details on this.

We can separate the dilatation operator $\mathcal{D}$ into a classical part $\mathcal{D}_0$ and quantum corrections $\delta\mathcal{D}$ as

$$\mathcal{D} = \mathcal{D}_0 + \delta\mathcal{D}\,. \tag{2.92}$$

The result for the two point function with (2.85) then gives us the form of the one-loop contribution to the dilatation operator. The Hamiltonian of the corresponding spin-chain is this contribution up to a factor of the coupling constant squared

$$\delta\mathcal{D} = \xi^2 H + \mathcal{O}(\xi^4)\,. \tag{2.93}$$

Above we have found the Hamiltonian density $\mathcal{H}$ of the closed sector given by ϕ_1 and ϕ_2. It is given by

$$\mathcal{H}_{k,k+1} = -2\delta^2_{I_k}\delta^1_{I_{k+1}}\delta_1^{J_k}\delta_2^{J_{k+1}}\,. \tag{2.94}$$

We will sometimes refer to this operator without the -2 as a chiral permutation operator, because it permutes the two vector spaces $\mathcal{V}_k$ and $\mathcal{V}_{k+1}$ if they are in chiral order as explained in chapter 3.

We will use with this definition of the dilatation operator, despite working in the framework of logarithmic CFTs. This turns out to be valid since in deriving the two-point functions in a logarithmic CFT, we have assumed that the action of D on an operator and its conjugate are identical. Since we are dealing with complex fields here, this does not hold and a redefinition of operator conjugation can both turn $\delta\mathcal{D}$ into $\xi^2 H$ and bring the correlators into the appropriate form for a logarithmic CFT.

In fact, if the above derivation leads to a non-diagonalisable H, then clearly the two-point function does not take the form given in (2.85). Let us assume that we obtain a non-diagonalisable H from the above procedure, with

$$HO_1 = EO_1 \text{ and } HO_2 = EO_2 + O_1\,. \tag{2.95}$$

Due to the chirality of the theory we have the opposite effect on the hermitian conjugates, i.e.,

$$HO_2^\dagger = EO_2^\dagger \text{ and } HO_1^\dagger = EO_1^\dagger + O_2^\dagger\,. \tag{2.96}$$

Then, it is easy to check that their correlation functions up to one loop take the form

$$\langle O_i^\dagger(x_1) O_j(x_2)\rangle = \begin{pmatrix} \frac{C}{|x_1-x_2|^{2\Delta}} & \frac{B\log(\mu^2(x_1-x_2)^2)}{|x_1-x_2|^{2\Delta_0}} \\ 0 & \frac{C}{|x_1-x_2|^{2\Delta}} \end{pmatrix}. \tag{2.97}$$

A simple relabelling of the operators $\bar{O}_1 = O_2^\dagger$ and $\bar{O}_2 = O_1^\dagger$ leads to the expected result in (2.46) to (2.49). The relabeling also makes sure that $\bar{O}_i$ and O_i transform identically under H.

We have now introduced the most important concepts relevant for understanding the research this thesis reports on. In the next two chapters, we report on the main results of our research. We will see how to bring large parts of the one-loop dilatation operator of the fishnet theory and the strongly-β-twisted theory or equivalently the spin-chain Hamiltonian in Jordan normal form.

Chapter 3

Spectrum: Methodology

In this chapter, we turn to diagonalising the one-loop dilatation operator of the strongly-twisted deformations of $\mathcal{N}=4$ SYM. At the start, we will derive the one-loop dilatation operator or equivalently the Hamiltonian of the corresponding spin-chain from the twisted but not scaled version. Afterwards, we discuss both the coordinate Bethe ansatz, the algebraic Bethe ansatz and the nested algebraic Bethe ansatz. All three ansätze lead to a set of equations whose solutions give the eigenvalues of the spin-chain Hamiltonian. In between, we discover several features of the Hamiltonian that simplify finding its eigenvalue, the most important of which is the notion of eclectic field content given in 3.3.2. In section 3.5, we suggest a possibility to start with a given set of twisted unscaled Bethe equations and systematically perform the scaling at the level of the equations. Most of this chapter is based on [1], however, significantly more details of the derivations are given. The Bethe equations of the largest non-eclectic sector at the end of the chapter are so far unpublished.

3.1 The Dilatation Operator of Strongly-Twisted $\mathcal{N}$=4 SYM

Our first task is to find the one-loop dilatation operator of the theories we want to investigate. The one-loop dilatation operator of unscaled twisted $\mathcal{N}=4$ SYM has been worked out in a series of papers [49, 48]. The Hamiltonian density of the corresponding spin-chain is

$$(\mathcal{H}^{\gamma})^{A'_n A'_{n+1}}_{A_n A_{n+1}} = \exp\left(\frac{-i}{2}\left((\mathbf{q}_{A_n})^T C \mathbf{q}_{A_{n+1}} + (\mathbf{q}_{A'_{n+1}})^T C \mathbf{q}_{A'_n}\right)\right)(\mathcal{H}^{\mathcal{N}=4})^{A'_n A'_{n+1}}_{A_n A_{n+1}}\,. \tag{3.1}$$

Here $\mathcal{H}^{\mathcal{N}=4}$ is the one-loop Hamiltonian density of the spin-chain of untwisted $\mathcal{N}$=4 [18], $\mathbf{q}_A$ are charge vectors given in table 2.1 and C is the twist matrix

$$C = \begin{pmatrix} 0 & -\gamma_3 & \gamma_2 \\ \gamma_3 & 0 & -\gamma_1 \\ -\gamma_2 & \gamma_1 & 0 \end{pmatrix}. \tag{3.2}$$

The vector space associated, with each spin-chain site is spanned by the basis $D^k\phi_i$, $D^k\phi_i^\dagger$, $D^k\psi_j$, $D^k\bar{\psi}_j$, $D^k\mathcal{F}$ and $D^k\tilde{\mathcal{F}}$, for $i=1,2,3$, $j=1,2,3,4$ and k a non-negative integer. Here $\mathcal{F}$ is the field strength tensor, D are covariant derivatives, which will become ordinary partial derivatives in the strong twisting limit, ϕ and ψ are the scalars and fermions respectively, and we have suppressed all space-time indices. We refer to the above basis vectors as letters. Consequently, spin-chain states, which are just tensor products of these letters are called words. Note that since we can have an arbitrary number of derivatives

acting on a single spin-chain site our vector space is infinite-dimensional already for a single site. Let us further denote the flavour of a letter by F, e.g., $F(\partial^k \phi_i) = i$.

Let us be more explicit about how $\mathcal{H}^{\mathcal{N}=4}$ acts on two neighbouring sites. The states of these sites correspond to sets of oscillators as described in section 2.2.1, $\mathcal{H}$ creates a linear combination of all possible distributions of the oscillators on the two sites. The coefficients of the states in the linear combination are given by[1]

$$c[n, n_{12}, n_{21}] = 2(-1)^{n_{12}n_{21}} \frac{\Gamma\left(\frac{1}{2}(n_{12}+n_{21})\right)\Gamma\left(1+\frac{1}{2}(n-n_{12}-n_{21})\right)}{\Gamma\left(1+\frac{n}{2}\right)}\,, \tag{3.3}$$

where n_{12} and n_{21} are the number of oscillators moving from one site to the other and n is the total number of oscillators. For the special case of $n_{12} = n_{21} = 0$, we have

$$c[n, 0, 0] = 2h\left(\frac{n}{2}\right)\,, \tag{3.4}$$

where h are the harmonic numbers.

The Hamiltonian density in (3.1) contributes at order g^2 in perturbation theory. In the strongly-twisted models, we interpret the order $\xi^2 = g^2 q^2$ as the one-loop correction. Hence, when we want to take the double scaling limit, we need to pull out a factor of q^2 from (3.1) to combine with the g^2 to give a ξ^2. Put differently, only the part diverging as q^2 in (3.1) survives the double-scaling limit and the remaining matrix elements go to zero.[2] To determine which part of (3.1) diverges like q^2, we first notice that both $(\mathbf{q}_{A_n})^T C \mathbf{q}_{A_{n+1}}$ and $(\mathbf{q}_{A'_{n+1}})^T C \mathbf{q}_{A'_n}$ are at most γ_i, and hence to obtain a factor of q^2 they both have to be equal to γ_i. This allows us to look at the incoming and outgoing letters separately. Which set of incoming and outgoing letters actually give us the required factor of q we then find by a case by case analysis. For example, if $A_n = \phi_1$ or $A_n = \psi_1$, then $A_{n+1} = \phi_3$, $A_{n+1} = \psi_3$, $A_{n+1} = \phi_2^\dagger$ or $A_{n+1} = \bar{\psi}_2$, with derivatives potentially acting on either A_n or A_{n+1}. Interestingly, we notice that the factor of q does only depend on the flavour of the field and not on the fermionic or bosonic nature, or whether there are derivatives acting on the spin-chain site.

This leads us naturally to the notion of chiral order, which we define as follows. First of all, two letters are never in chiral order, if either of them is $\partial^k F$, $\partial^k \psi_4$ or $\partial^k \bar{\psi}_4$. Assuming neither of the two letters are of the above type, then if both A_n and A_{n+1} are fields, or both of them are antifields, they are in chiral order if

$$F(A_{n+1}) = F(A_n) - 1 \mod 3\,. \tag{3.5}$$

If one of A_n and A_{n+1} is a field and the other is an antifield, they are in chiral order, if

$$F(A_{n+1}) = F(A_n) + 1 \mod 3\,. \tag{3.6}$$

Finally we also define antichiral order as exactly the opposite order, i.e., A_n and A_{n+1} are in antichiral order, if after swapping them they are in chiral order.[3]

We have defined this notion of chiral and antichiral order since it allows us to write down the Hamiltonian densities of the spin-chains corresponding to the strongly-twisted models. In particular we see, that in (3.1) to obtain the appropriate factors of q^2 we require A_n and A_{n+1} to be in chiral order and A'_n and A'_{n+1} to be in antichiral order. Due to the

[1] The extra factor of two appearing in (3.3) as compared to [18] is due to a different convention in the coupling constant.

[2] Nothing in (3.1) diverges faster than q^2.

[3] Although this might seem very technical, it becomes intuitive very quickly.

nature of $\mathcal{H}^{\mathcal{N}=4}$ this can only happen at one-loop if the flavour is simply getting exchanged when acting with $\mathcal{H}$. In order to write down an explicit formula for the Hamiltonians of the strongly-twisted models, we define operators P^- and P^+ that project onto the subspaces of chirally ordered and antichirally ordered pairs respectively. Then we have for the strongly-β-twisted theory the following one-loop Hamiltonian density

$$\mathcal{H}^{\mathrm{s}\beta\mathrm{t}}_{n,n+1} = P^+_{n,n+1}\mathcal{H}^{\mathcal{N}=4}_{n,n+1}P^-_{n,n+1}\,. \tag{3.7}$$

When moving from the β-twisted to the more general γ_i-twisted theory, the only change appearing is an additional factor c multiplying the Hamiltonian density

$$(\mathcal{H}^{\mathrm{s}\gamma\mathrm{t}}_{n,n+1})^{A'_nA'_{n+1}}_{A_nA_{n+1}} = c(a_1,a_2,a_3)\,(\mathcal{H}^{\mathrm{s}\beta\mathrm{t}}_{n,n+1})^{A'_nA'_{n+1}}_{A_nA_{n+1}}\,. \tag{3.8}$$

The factor c depends solely on the ratio of the different couplings ξ_i. We parameterise these couplings with respect to some reference coupling ξ as

$$\xi_i = \xi a_i\,. \tag{3.9}$$

As we saw before, the twist factor of (3.1) factorises into a piece depending on the incoming fields and one depending on the outgoing fields. Hence, the additional prefactor c of the Hamiltonian density also factorises

$$c(a_1,a_2,a_3) = c_{\mathrm{in}}(a_1,a_2,a_3)c_{\mathrm{out}}(a_1,a_2,a_3) \tag{3.10}$$

If the incoming particles are both scalars, the corresponding prefactor is $c_{\mathrm{in}} = a_i$, where the i refers to the absent flavor. If the incoming particles are both fermions, we have $c_{\mathrm{in}} = \sqrt{a_ia_j}$, with i and j referring to the flavours of the fermions. Finally, if the incoming particles are one fermion and one scalar we find $c_{\mathrm{in}} = \sqrt{a_ia_j}$, with i referring to the flavour of the fermion and j referring to the absent flavour. The same rules determine c_{out}. The above discussion includes the special case, where two of the a_i are zero, which yields the fishnet theory. In this case, the exchange of a fermion and a scalar is forbidden, since c_{in} contains two different a_i. Similarly, in this case, the exchange of two fermions is forbidden, because for the fermions to be in chiral order, they need to be of different flavour and consequently c_{in} contains two different a_i. The only allowed exchanges are between two remaining scalar fields and their conjugates. For example, if $a_1 = a_2 = 0$, only exchanges between ϕ_1, ϕ_2, $\phi_1^\dagger$ and $\phi_2^\dagger$ are allowed. This is perfectly in line with our expectations because, as we saw previously, all other fields decouple in the fishnet limit.

The above derivation of the one-loop dilatation operator, or equivalently the corresponding spin-chain Hamiltonians is correct. However, it goes slightly against the general spirit of this thesis. Our main goal is to take the strongly-twisted theories, as defined by the Lagrangian densities and in a bottom-up approach find and diagonalise the dilatation operator. According to this argument, one has to calculate the one-loop Feynman diagrams corresponding to two-point functions in these theories. However, the Feynman diagrams appearing in these calculations are just a subset of those appearing in the calculation of the two-point functions of the untwisted theories, which have been calculated many times. We will not repeat the calculation here, see however section 2.3.2. The alternative approach discussed above starts from the Hamiltonian of unscaled twisted $\mathcal{N} = 4$ SYM. Generally, a problem arises when one attempts to scale results of the unscaled twisted theories. Explicitly, there are many variables for which we do not know the twist-dependence. An example are the Bethe roots arising the Bethe ansatz as explained in section 3.2. Many ways of scaling these variables are possible, and we will have to guess the correct one.

There might potentially even be several correct scaling behaviours. However, concerning the dilatation operator resulting from the above argument, there are no free variables. Thus, this second approach will yield the correct one-loop dilatation operator.

Just like the Hamiltonian of the $\mathcal{N} = 4$ SYM spin-chain, the Hamiltonian of our spin-chain contains invariant subspaces, meaning subspaces that get mapped onto themselves under the action of the Hamiltonian. This might happen, for example, if we limit the field content of the operators we consider. For example, when one acts with the Hamiltonian on a single-trace operator containing only ϕ_1 and ϕ_2, one again obtains an operator made of ϕ_1 and ϕ_2, albeit differently arranged. Invariant subspaces with limited field content are called closed sectors and can either be one-loop closed if they are invariant under the one-loop Hamiltonian or all-loop closed if they are invariant under the full Hamiltonian. These closed sectors and their Hamiltonians tend to be much simpler than the full model and offer a good starting point for our investigation of the spectrum. Thus, we will look at these sectors one by one for the rest of this chapter and present the methods to diagonalise the Hamiltonian restricted to them.

3.2 The Two-Scalar Sector

We will now discuss the simplest sector of the strongly-twisted models, which consists of operators containing only two scalar fields, no derivatives and no fermions. If we were to use a field and its conjugate like ϕ_1 and $\phi_1^\dagger$, the fields would never be in chiral order, as described in section 3.1. Consequently, the Hamiltonian would be identically zero. All other choices are equivalent and for our discussion we use the ϕ_1 and ϕ_2 fields. We immediately move to the spin-chain picture. The vector space of a single spin-chain site is two-dimensional, and we call ϕ_1 up spins and ϕ_2 down spins. In this sector, the Hamiltonian is particularly simple, it is only a chiral permutation operator [54], i.e., it scans the spin-chain until it finds two fields in chiral order and swaps them

$$\mathcal{H}_{i,i+1} = -2\sigma_i^+\sigma_{i+1}^-\,, \tag{3.11}$$

where σ^+ annihilates up spins and flips down spins, while σ^- annihilates down spins and flips up spins. We impose periodic boundary conditions $L + 1 = 1$.

In the following sections, we will diagonalise the Hamiltonian corresponding to the density given in (3.11) using three different methods. However, notice that the Hamiltonian is explicitly non-hermitian and hence we have no reason to expect the Hamiltonian to be diagonalisable. In fact, if we did not impose periodic boundary conditions, the above Hamiltonian would be triangular with zeros on the diagonal and would indeed not be diagonalisable. Introducing periodic boundary conditions does in fact yield a diagonalisable Hamiltonian, as we will explicitly see in the following. However, this is the only non-trivial[4] diagonalisable sector in the fishnet and strongly-β-twisted theories and we will have to keep this issue of non-hermiticity of the Hamiltonian in mind, when we discuss other sectors.

3.2.1 Coordinate Bethe Ansatz

We will now give a quick overview of the basics of the coordinate Bethe ansatz, abbreviated CBA, invented by Hans Bethe to diagonalise the XXX-Heisenberg spin-chain [16]. There are many review papers considering the CBA for the Heisenberg spin-chain see for example [79]. For two reasons, we will discuss the Bethe ansatz for the example of the two-scalar

[4]Some sectors are just annihilated by the Hamiltonian, i.e., $H = 0$, which is diagonal, but not very interesting.

sector of the strongly-twisted models. Firstly, this model offers perhaps the simplest Hamiltonian possible. Secondly, as far as we know, we were the first to actually use a CBA to diagonalise the Hamiltonian.

The first idea of the Bethe ansatz is to interpret one type of spin-chain site state as a vacuum, let us say spin-up, and describe the spin-down states as quasiparticles, called magnons, travelling along the spin-chain. The number of down spins is fixed under the action of (3.11), since the Hamiltonian always flips two spins, one of which is an up-spin and the other a down-spin. This naturally splits all the possible states of the complete spin-chain into groups labelled by the number of down spins M. If we now look at a state with a single magnon at position i denoted $|i\rangle$, acting once with the Hamiltonian will push this magnon to the next site $i+1$. If we want an eigenstate of the Hamiltonian with one magnon, we hence need a superposition of the states, where the magnon is at positions one through L, related by the eigenvalue E of the Hamiltonian:

$$|\Psi\rangle = \sum_{l=1}^{L} \left(\frac{E}{-2}\right)^{-l} |l\rangle = \sum_{l=1}^{L} e^{ipl} |l\rangle \ , \tag{3.12}$$

where in the last step we have defined the spin-chain momentum p. Finally, when we impose periodic boundary conditions we obtain a condition on p or in the one magnon case equivalently E, corresponding to the Bethe equation, in this case, $\left(\frac{E}{-2}\right)^{L} = 1$. If we include a second magnon, the two excitations can have different spin-chain momenta and the general ansatz is

$$|\Psi\rangle = \sum_{l=2}^{L} \sum_{k=1}^{l-1} (\exp(i(kp_1 + lp_2)) + S(p_1, p_2) \exp(i(kp_2 + lp_1))) |k, l\rangle \ , \tag{3.13}$$

where S is a complex function of the ps called the S-matrix and $|k, l\rangle$ is the spin-chain state with excitations at k and l. Looking at states, for which the excitations are far apart gives the eigenvalue of the Hamiltonian as

$$E = -2(e^{-ip_1} + e^{-ip_2}) \, . \tag{3.14}$$

The S-matrix can be determined by looking at the action of the Hamiltonian on the state $|k-1, k+1\rangle$ and comparing to the coefficient of $|k, k+1\rangle$ in (3.13), which yields $S = -1$ in our case. For an infinitely long chain, this would already yield the eigenstates of the Hamiltonian, however, there is an artificial break between spin-chain sites L and one. To impose periodic boundary conditions we compare the part $\exp(i(p_1 + lp_2)) |1, l\rangle$ of (3.13) with $S(p_1, p_2) \exp(i(lp_2 + Lp_1))) |l, L\rangle$ from the same linear combination. Setting the coefficient of the first equal to the coefficient of the second times e^{ip_1} guarantees that the two artificial ends of the spin-chain at spin-chain sites one and L match and yields one of the Bethe equations

$$e^{ip_1 L} = S(p_1, p_2) = -1 \, . \tag{3.15}$$

This is usually referred to in the literature as taking the magnon once around the spin-chain.[5] There is nothing special about p_1 and hence we also have a second equation, which is just the same condition for p_2, the appropriate coefficients to compare are

[5] Since I found this widespread language of taking particles around the spin-chain confusing, I want to make clear that the Bethe equation just comes from glueing the ends of the spin-chain together. However, if the image of particles being taken around the spin-chain helps this is obviously a perfectly fine way of thinking about it.

$S(p_1,p_2)\exp(i(1p_2+lp_1))\,|1,l\rangle$ and $\exp(i(lp_1+Lp_2))\,|l,L\rangle$. If the Bethe equations are fulfilled (3.13) are the two-magnon eigenstates of the Hamiltonian.

The final step of the Bethe ansatz is the generalisation of equation (3.13) to M magnons. The idea is again that when the excitations are far apart, they are just plane waves. When the excitations are on neighbouring sites, an S-matrix is introduced to match the plane waves. The general ansatz for an M magnon eigenstate is

$$|\Psi\rangle = \sum_{n_1<n_2<\cdots<n_M}^{L} \sum_{\sigma\in S_M} S_\sigma e^{n_1 p_{\sigma(1)}+\cdots+n_M p_{\sigma(M)}}\,|n_1,\ldots,n_M\rangle\,. \tag{3.16}$$

The state $|n_1,\ldots,n_M\rangle$ is the state with down spins at positions n_1, n_2, etc. We sum over all the positions these excitations can be in, with a given coefficient. Those coefficients are a sum over permutations σ, which assign a momentum p_k to each excitation. If the Hamiltonian pushes one of the excitations, which is well separated from the other excitations at position i to $i+1$, we can compare coefficients before and after acting with H and find that the energy has to be

$$E = -2\sum_{k=1}^{M} e^{-ip_k}\,. \tag{3.17}$$

The S_σ in (3.16) is a product of the two-particle S-matrices, such that exchanging the particles with the S-matrices puts them in the order given by σ. Different decompositions of S_σ agree since the two-particle S-matrices fulfil a Yang-Baxter equation. Finally, when we close the chain we get one coondition for each p_k of the form

$$e^{ip_k L} = \prod_{j\neq k}^{M} S(p_k,p_l) = (-1)^{M-1}, \tag{3.18}$$

which are the Bethe equations for the two-scalar model. If we are interested in the correlation functions of single trace operators, we only want cyclically invariant eigenstates, which is easily imposed via the zero-momentum condition

$$\prod_{k=1}^{M} e^{ip_k} = 1\,. \tag{3.19}$$

This equation is just the condition that the spin-chain state is invariant under a shift of all excitations by an integer number of spin-chain sites.

In the two scalar sector, the Bethe equations are extremely simple. If we are only interested in the Eigenvalues of the Hamiltonian, i.e., the one-loop anomalous dimensions, we are already done. They are simply given by the sum of roots of unity or negative unity, with one root for each excitation. In general the Bethe equations are coupled polynomial equations of degree up to $L+M$, and hence, usually, they are not analytically solvable. The main advantage of the Bethe ansatz for general models becomes apparent for large spin-chain length L. The Bethe ansatz reduces the problem to solving polynomial equations of degree $L+M$, instead of diagonalisation of a 2^L matrix. Thus in particular for small M the Bethe ansatz often simplifies the problem significantly and at least gives a numerical handle.

3.2.2 Algebraic Bethe Ansatz

In the last subsection, we saw one way of diagonalising the Hamiltonian of the two-scalar model by making an explicit ansatz, namely the coordinate Bethe ansatz (CBA). In this

subsection, we turn to the second very popular form of the Bethe ansatz going under the name of algebraic Bethe ansatz (ABA) or quantum inverse scattering method (QISM). This second version of the Bethe ansatz has emerged in the 70's, mainly developed by the 'Leningrad school'. The most popular review of this version of the ABA is by Fadeev [80], one of the lead developers of the method, which is expanded upon in [81]. References [78] and [79] give more accessible introductions, while [82] is more advanced.

The main object of the ABA is an R-matrix, which acts in the tensor product of two vector spaces, which are of the form of spin-chain sites. The R-matrix for the unscaled twisted models is known [47]

$$R(u) = \frac{1}{u+i}\left(uf(q)I + iP\right), \tag{3.20}$$

where I is the identity, u the so called spectral parameter and $f(q)$ is the twist factor, which is at most q^2. This value is obtained exactly when the incoming vector spaces are in chiral order, and, for the case of the fishnet model, no decoupled fields appear. Since u is the only free parameter in the R-matrix, it has to scale as $u = i\lambda q^{-2}$, and we find

$$R(u) = \lambda\mathcal{P}^- + P\,, \tag{3.21}$$

where $\mathcal{P}^-$ is the chiral projection operator defined in the last section and P is the permutation operator, simply exchanging the states of sites one and two. In standard matrix notation we have

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.22}$$

This matrix fulfils the important Yang-Baxter equation

$$R_{a,b}(u-v)R_{a,c}(u)R_{b,c}(v) = R_{b,c}(v)R_{a,c}(u)R_{a,b}(u-v) \tag{3.23}$$

Here, each R-matrix is taken to act in the threefold tensor product of three vector spaces a, b and c, acting as described above in two of them and as the identity in the third. The Yang-Baxter equation (3.23) is at the heart of the algebraic Bethe ansatz leading to the mathematical relations we will need in the following. We introduce an auxiliary space a, which is $\mathbb{C}^2$ just like a spin chain site. Then, we can define a monodromy matrix acting in the tensor product of the auxilliary space and the Hilbert space of our spin chain

$$M(\lambda) = R_{a,N}(\lambda)R_{a,N-1}(\lambda)\cdots R_{a,1}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{3.24}$$

On the right hand side we explicitly wrote the monodromy as a matrix in the auxilliary space, with entries being operators acting on the Hilbert space of the spin chain. Taking the trace of the monodromy we obtain the transfer matrix

$$T(\lambda) = A(\lambda) + D(\lambda). \tag{3.25}$$

It is easy to check that the following expressions for the transfer matrix and its derivative at $\lambda = 0$ hold

$$T(0) = P_{1,2}P_{2,3}\cdots P_{L-1,L} \tag{3.26}$$

$$\frac{d}{d\lambda}T(\lambda)\Big|_{\lambda=0} = \sum_{n=1}^{L} P_{1,2}\cdots P_{n-2,n-1}P_{n-1,n+1}P_{n+1,n+2}\cdots P_{L-1,L}\mathcal{P}^-_{n-1,n} \tag{3.27}$$

$$-\frac{H}{2} = T^{-1}(0)\frac{d}{d\lambda}T(\lambda)\Big|_{\lambda=0}, \tag{3.28}$$

where H is the spin chain Hamiltonian. Here indices stand for vector spaces from the tensor product in which operators do not act trivially. From the last equation, we see that instead of diagonalising the Hamiltonian, we can diagonalise the transfer matrix. We already know two states, which diagonalise the Hamiltonian already, namely the ones where all the spins are aligned and indeed calling the state with all spins up Ω we find

$$A(\lambda)\Omega = \Omega \tag{3.29}$$

$$D(\lambda)\Omega = \lambda^L \Omega\,. \tag{3.30}$$

When we now act with $B(\lambda)$ on Ω, we will create superpositions of states with some spins flipped. To find out whether a state of the form

$$\Psi = \prod_{i=1}^{M} B(\lambda_i)\Omega \tag{3.31}$$

is an eigenstate of the transfer matrix we need to commute the transfer matrix past the Bs. This is where the Yang-Baxter equation (3.23) becomes crucial. From the Yang-Baxter equation it is easy to show that the monodromies fulfil

$$R_{a_1,a_2}(\lambda-\mu)M_{a_1}(\lambda)M_{a_2}(\mu) = M_{a_2}(\mu)M_{a_1}(\lambda)R_{a_1,a_2}(\lambda-\mu). \tag{3.32}$$

This is a 4x4 matrix equation, three entries of which are of particular interest to us, they are

$$[B(\lambda), B(\mu)] = 0 \tag{3.33}$$

$$A(\lambda)B(\mu) = \frac{1}{\lambda-\mu}B(\lambda)A(\mu) - \frac{1}{\lambda-\mu}B(\mu)A(\lambda) \tag{3.34}$$

$$D(\lambda)B(\mu) = \frac{1}{\lambda-\mu}B(\mu)A(\lambda) - \frac{1}{\lambda-\mu}B(\lambda)A(\mu)\,. \tag{3.35}$$

This gives us the relations we need to permute the transfer matrix past the Bs in (3.31). We start by pushing the A-operator past the Bs

$$A(\lambda)\Psi = \prod_{k}^{M} -\frac{1}{\lambda-\lambda_k}\Psi + \sum_{k=1}^{M}\frac{1}{\lambda-\lambda_k}\prod_{j\neq k}^{M} -\frac{1}{\lambda_k-\lambda_j}B(\lambda_1)\cdots B(\lambda_{k-1})B(\lambda_{k+1})\cdots B(\lambda_M)B(\lambda)\Omega\,. \tag{3.36}$$

The coefficient for the second term is easy to determine for $k=1$, but since the Bs all commute the form has to be the same for all k. The second term makes Ψ not an eigenstate of A and is thus usually referred to as 'unwanted'. Acting with D we obtain a similar term and the unwanted terms can cancel for specific values of the λ_k. This happens, when

$$\left(\prod_{k\neq i}^{M}\frac{1}{\lambda_i-\lambda_k}\right)\left((-1)^{M-1}-\lambda_i^L\right) = 0 \quad \Longrightarrow \quad \lambda_i^L = (-1)^{M-1}. \tag{3.37}$$

These equations should look familiar, they are just the Bethe equations for this model (3.18), with $\lambda_k = e^{-ip_k}$. From the eigenvalue of the transfer matrix we can also calculate the energy which turns out to be

$$E = -2\sum_{k=1}^{M}\frac{1}{\lambda_k} \tag{3.38}$$

in agreement with (3.17). Note that, since the λ_ks are roots of (negative) unity, their inverses also form a valid solution set.

3.2.3 Jordan-Wigner Transformation

After explaining the machinery of the two different types of the Bethe ansatz, we will now turn to an explicit way to diagonalise the Hamiltonian of our simple example. The method, consisting out of two consecutive transformations, has been used more than 50 years ago to diagonalise the XY-model [83]. In particular we will give a two step transformation, which will transform the Hamiltonian into

$$H = \sum_{k=1}^{M} E_k \eta_k^\dagger \eta_k \,, \tag{3.39}$$

where η_k are fermionic raising and lowering operators. Upon closer inspection, we see that our Hamiltonian (3.11) is actually already not that far from the desired form. There are only two main differences. Firstly, while $\sigma_k^\pm$ already fulfil the desired anticommutation relations on the same site, they commute at different sites, i.e.

$$\{\sigma_j^+, \sigma_j^-\} = 1 \quad \sigma_i^{+2} = \sigma_i^{-2} = 0 \tag{3.40}$$

$$[\sigma_j^\pm, \sigma_k^\pm] = 0 \,, \text{for j} \neq \text{k} \,. \tag{3.41}$$

Secondly, the Hamiltonian density has the two sigma-matrices acting on neighbouring sites and not on the same site. The first issue we can transform away by a Jordan-Wigner transformation, i.e. by introducing fermionic raising and lowering operators as

$$c_k^\dagger = \sigma_k^- \exp\left(-i\pi \sum_{j=1}^{K-1} \sigma_j^+ \sigma_j^+\right) \,, \quad c_k = \exp\left(i\pi \sum_{j=1}^{K-1} \sigma_j^+ \sigma_j^-\right) \sigma_k^+ \,. \tag{3.42}$$

These c-operators fulfil (3.41) except with the commutator replaced by an anticommutator. The Hamiltonian becomes

$$H = 2 \sum_{j=1}^{L-1} c_k^\dagger c_{k+1} + 2(-1)^{M-1} c_L^\dagger c_1 \,, \tag{3.43}$$

where M is the number of down spins as before. In a second step, we now write the c operators as a plane wave of another set of operators η, which again fulfil canonical anticommutation relations. Since the Hamiltonian (3.43) does not change the number of downspins we can make a distinction depending on whether we have an even or odd number of them. We use

$$c_k = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{ijk2\pi/L} \eta_j \,, \qquad \text{for } M \text{ odd} \tag{3.44}$$

$$c_k = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} e^{i(2j+1)k\pi/L} \eta_j \,, \qquad \text{for } M \text{ even} \,, \tag{3.45}$$

where we have not written out the corresponding hermitian conjugates. The η_k fullfill the desired anticommutation relations for fermionic raising and lowering operators, i.e.

$$\{\eta_k^\dagger, \eta_k\} = 1 \quad \eta_k^{\dagger 2} = \eta_k^2 = 0 \,, \tag{3.46}$$

and they all anticommute for different indices. It is relatively easy to show that in terms of these operators the Hamiltonian is indeed given by (3.39), with

$$E_k = 2e^{ik2\pi/L} \,, \quad \text{for M odd and} \tag{3.47}$$

$$E_k = 2e^{i(k+1)2\pi/L} \,, \quad \text{for M even.} \tag{3.48}$$

The only subtle part of the calculation is that for even M the second transformation is *not* periodic in k with period L and it is important to always use c_1 instead of c_{L+1}.

We have now diagonalised the Hamiltonian of the simplest sector that appears in the strongly-twisted models in three different ways. For non-hermitian operators, it is not guaranteed that a sufficient number of eigenvectors can be found to diagonalise the matrix. The only way to show that the Hamiltonian is diagonalisable with the help of a Bethe ansatz, is to count the solutions. In fact, for the two-scalar sector, we can do so. For a spin-chain of length L with M magnons a diagonalisable Hamiltonian will have $\binom{L}{M}$ eigenstates. Indeed we choose exactly M Bethe roots out of L roots of (negative) unity, so we find all eigenstates with the Bethe ansatz. The Jordan-Wigner transformation we have described brings the Hamiltonian in explicitly diagonal form, so it shows in the most straightforward way that it is actually diagonalisable. The two-scalar sector is already interesting in itself, and we will later discuss the spectrum and the solutions to the Bethe equations for this sector, but for now, we turn to more complicated sectors.

3.3 Adding a Third Scalar

After diagonalising the two-scalar Hamiltonian, we now want to add additional fields to the operators we consider. Starting with ϕ_1 and ϕ_2, we have four different scalars we could possibly include, namely $\phi_1^\dagger$, $\phi_2^\dagger$, ϕ_3 and $\phi_3^\dagger$. As it turns out, it makes a huge difference which scalar we choose. $\phi_1^\dagger$ and $\phi_2^\dagger$ we will only briefly mention. Intuitively it might feel most natural to add ϕ_3 excitations, however, as we will discuss in some detail later on this is a 'bad' choice. The most easily added scalar is the $\phi_3^\dagger$, which we call the 'good' case[6].

3.3.1 The Good

In this subsection we consider single trace operators built from the fields ϕ_1, ϕ_2 and $\phi_3^\dagger$ for example $O = \mathrm{tr}(\phi_1^2\phi_2\phi_3^\dagger)$. This means that in the spin chain picture, we now have a three-dimensional vector space associated with each spin chain site. We identify

$$\phi_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad \phi_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \phi_3^\dagger = \begin{pmatrix}0\\1\\0\end{pmatrix}. \tag{3.49}$$

This convention might appear strange at first sight, but it enables us to apply the ABA in its standard form, as we will see later. We consider this sector for now only in the strongly-β-twisted theory, where all couplings are the same. In terms of the above representation of the fields we can write the Hamiltonian density as

$$\begin{aligned}\mathcal{H}_{n,n+1} = &\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}_n \otimes \begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}_{n+1} + \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}_n \otimes \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}_{n+1}\\ &+ \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}_n \otimes \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}_{n+1}.\end{aligned} \tag{3.50}$$

[6]Technically this is not the only case, which we will call 'good'. There are a total of twenty possibilities to choose three fields out of six, six of which are good. More details on this will be given when we explain why some choices are 'bad'.

To diagonalise the corresponding Hamiltonian, we use an algebraic Bethe Ansatz, the basics of which have been described in the last section. However, since we have now two different types of excitations, we need a nested version of the ABA, which we will describe in the following. The literature of nested ABAs is scarce; therefore we will try to give enough details to make the presentation self contained. A review, which unfortunately excludes the cancellation of the unwanted terms can be found in [84]. The cancellation of the unwanted terms is, for example, described in [85].

The nested ABA is still based on the R-matrix, which is still of the form given in (3.21). However, it now acts in the tensor product of two three-dimensional vector spaces and is hence a nine by nine matrix. We obtain the needed commutation relations between the matrix elements of the monodromy, T_{ab}, from the RTT-relations (3.32). We find, with $a, b, c, d, e \in \{2, 3\}$,

$$T_1^1(\lambda)T_1^a(\mu) = \frac{1}{\lambda-\mu}T_1^a(\lambda)T_1^1(\mu) - \frac{1}{\lambda-\mu}T_1^a(\mu)T_1^1(\lambda) \tag{3.51}$$

$$T_1^a(\lambda)T_1^b(\mu) = R_{dc}^{ab}(\lambda-\mu)T_1^c(\mu)T_1^d(\lambda) \tag{3.52}$$

$$T_a^b(\lambda)T_1^c(\mu) = \frac{-1}{\lambda-\mu}T_1^b(\lambda)T_a^c(\mu) + \frac{1}{\lambda-\mu}R_{ed}^{bc}(\lambda-\mu)T_1^d(\mu)T_a^e(\lambda), \tag{3.53}$$

where we have indicated the terms, which will produce unwanted terms in red. The components of the R-matrix appearing here correspond only to the second, i.e., $\phi_3^\dagger$ and the third, i.e., ϕ_2 basis-vectors. If we denote rows (incoming states) by lower indices and colums (outgoing states) by upper indices we can write the R-matrix in terms of Kronecker-deltas as

$$R_{ab}^{cd}(\lambda-\mu) = \delta_a^d\delta_b^c + (\lambda-\mu)\delta_a^2\delta_b^3\delta_2^c\delta_3^d\,, \tag{3.54}$$

which is just a different way of writing (3.22). Here the indices 2 and 3 stand for $\phi_3^\dagger$ and ϕ_2 respectively in the same convention as in (3.49). Had we used the more natural flipped convention, the R-matrix appearing would contain the antichiral projection operator, for which the ABA does not work in its standard form. The fact that the R-matrix of the two-scalar model appears in the commutation relations for the three-scalar model will yield the transfer matrix of the two-scalar model in solving the more advanced three-scalar model, which gives the method its name of nested Bethe Ansatz. There is a two-scalar spin-chain nested in the three-scalar spin-chain. We will make use of the following identity concerning the R-matrix, which is relatively easy to check explicitly

$$R_{ab}^{cd}(\lambda-\mu)R_{dc}^{ef}(\mu-\lambda) = \delta_a^f\delta_b^e\,. \tag{3.55}$$

While the technical details of the three scalar model are, as we will see, significantly more involved than for our two-scalar example, the basic idea is still the same. We are looking for eigenstates $|\Psi\rangle$ of the transfer matrix which are a linear combination of M $T_1^{a_j}(\lambda_j)$ operators acting on a vacuum $|\Omega\rangle$, which consists of only ϕ_1s, i.e., we are looking for linear combinations of states of the form

$$|\Psi\rangle = T_1^{a_1}(\lambda_1)\cdots T_1^{a_M}(\lambda_M)\,|\Omega\rangle\,. \tag{3.56}$$

We now use the commutation relations (3.51)-(3.53) to move T_1^1, T_2^2 and T_3^3 past the $T_1^{a_j}$. In analogy to the two-scalar model we get a wanted part, which is

$$(T_1^1(\lambda)\,|\Psi\rangle)_{\text{wanted}} = \left(\prod_{j=1}^{M}\frac{-1}{\lambda-\lambda_j}\right)|\Psi\rangle \tag{3.57}$$

for T_1^1. Acting with T_2^2 and T_3^3 also gives a wanted part, which now looks more complicated, i.e.,

$$\begin{aligned}(T_a^a(\lambda)\,|\Psi\rangle)_{\text{wanted}} =& R^{a\ a_1}_{e_1 d_1}(\lambda-\lambda_1)R^{e_1 a_2}_{e_2 d_2}(\lambda-\lambda_2)\cdots R^{e_{M-2}a_{M-1}}_{e_{M-1}d_{M-1}}(\lambda-\lambda_{M-1})R^{e_{M-1}a_M}_{a\quad d_M}(\lambda-\lambda_M)\\ &\times\lambda^L\left(\prod_{j=1}^{M}\frac{1}{\lambda-\lambda_j}\right)T_1^{d_1}(\lambda_1)\cdots T_1^{d_M}(\lambda_M)\,|\Omega\rangle\\ =&\lambda^L\left(\prod_{j=1}^{M}\frac{1}{\lambda-\lambda_j}\right)\mathcal{T}^{\{a_j\}}_{\{d_j\}}(\lambda,\{\lambda_j\})T_1^{d_1}(\lambda_1)\cdots T_1^{d_M}(\lambda_M)\,|\Omega\rangle\,,\end{aligned} \tag{3.58}$$

where we have defined $\mathcal{T}$ as the product of R-matrices appearing. $\mathcal{T}$ is the transfer matrix of the two-scalar model, however with inhomogeneities λ_k. Inhomogeneities is the name given to the λ_k, which are subtracted from λ in the argument of the individual R-matrices forming the transfer matrix.

We want a linear combination of states of the form of $|\Psi\rangle$ to be an eigenstate of $T_1^1+T_2^2+T_3^3$. Ignoring the unwanted part for the moment this is achieved, if the linear combination of $T_1^{d_1}\cdots T_1^{d_M}$ is an eigenstates of $\mathcal{T}^{\{a_j\}}_{\{d_j\}}(\lambda,\{\lambda_j\})$. This lower rank transfer matrix acts on the flavour part of our excitations, i.e., it acts on a two-scalar spin-chain given by the flavours a_j of the $T_1^{a_j}$ operators. Thus, we now turn to the diagonalisation of the lower dimensional transfer matrix. Luckily, we already know how to diagonalise this transfer matrix without inhomogeneities from section 3.2.2. The inhomogeneities do not affect the commutation relations as is easy to show. However, they affect the eigenvalues of the diagonal elements of the monodromy corresponding to the vacuum. As we have seen before we need to choose a vacuum to apply the ABA and we also have to choose a separate vacuum now to diagonalise this two-scalar transfer matrix. Choosing $\phi_3^\dagger$ as our vacuum $|0\rangle$ at this level of nesting, we have

$$A(\lambda)\,|0\rangle=|0\rangle\quad\text{and}\quad \mathrm{D}(\lambda)\,|0\rangle=\left(\prod_{\mathrm{j}=1}^{\mathrm{M}}(\lambda-\lambda_{\mathrm{j}})\right)|0\rangle\,. \tag{3.59}$$

As before we now create excited states by acting with $B(r_k)$ on $|0\rangle$. Cancellation of the unwanted terms now requires

$$\begin{aligned}&\frac{-1}{\lambda-r_k}\prod_{j\neq k}^{K}\frac{1}{r_k-r_j}\prod_{j=1}^{M}(r_k-\lambda_j)+\frac{1}{\lambda-r_k}\prod_{j\neq k}^{K}\frac{-1}{r_k-r_j}=0\\ \Longrightarrow&\prod_{j=1}^{M}(r_k-\lambda_j)=(-1)^{K-1}.\end{aligned} \tag{3.60}$$

The eigenvalue of the rank two transfer matrix is simply

$$\Lambda=\prod_{j=1}^{K}\frac{1}{\lambda-r_j}\left((\lambda-\lambda_k)^L+(-1)^K\right), \tag{3.61}$$

which is determined using (3.59).

When commuting the diagonal elements of the monodromy past the $T_1^{a_j}$ we also pick up unwanted terms on the original level. These terms are defined by containing a creation operator evaluated at the spectral parameter λ instead of λ_k. We will impose that these

terms cancel, which will lead to a second set of Bethe equations. To find the unwanted term, with $T_1^b(\lambda_k)$ missing we first pull the kth creation operator to the front

$$\begin{aligned}|\Psi\rangle =&R^{a_{k-1}a_k}_{b_{k-1}c_{k-1}}(\lambda_{k-1}-\lambda_k)R^{a_{k-2}c_{k-1}}_{b_{k-2}c_{k-2}}(\lambda_{k-2}-\lambda_k)\cdots R^{a_2c_3}_{b_2c_2}(\lambda_2-\lambda_k)R^{a_1c_2}_{b_1b_k}(\lambda_2-\lambda_k)*\\ &T_1^{b_k}(\lambda_k)T_1^{b_1}(\lambda_1)\cdots T_1^{b_{k-1}}(\lambda_{k-1})T_1^{a_{k+1}}(\lambda_{k+1})\cdots T_1^{a_M}(\lambda_M)\,|\Omega\rangle\,.\end{aligned} \tag{3.62}$$

We can now compute the unwanted terms by pulling $T_1^1(\lambda)$ and $T_a^a(\lambda)$ past the raising operators. We use the red terms in (3.51) and (3.53) when commuting past the first raising operator and the other terms when commuting past the others. We find

$$\begin{aligned}(T_1^1(\lambda)\,|\Psi\rangle)_{\text{uw.(k)}} =& R^{a_{k-1}a_k}_{b_{k-1}c_{k-1}}(\lambda_{k-1}-\lambda_k)R^{a_{k-2}c_{k-1}}_{b_{k-2}c_{k-2}}(\lambda_{k-2}-\lambda_k)\cdots R^{a_2c_3}_{b_2c_2}(\lambda_2-\lambda_k)R^{a_1c_2}_{b_1b_k}(\lambda_2-\lambda_k)*\\ &\frac{1}{\lambda-\lambda_k}\left(\prod_{j\neq k}^{M}\frac{-1}{\lambda_k-\lambda_j}\right)T_1^{b_k}(\lambda)T_1^{b_1}(\lambda_1)\cdots T_1^{b_{k-1}}(\lambda_{k-1})T_1^{a_{k+1}}(\lambda_{k+1})\cdots T_1^{a_M}(\lambda_M)\,|\Omega\rangle\end{aligned} \tag{3.63}$$

for $T_1^1(\lambda)$. The string of R-matrices and raising operators is nearly invariant under pulling T_1^1 through, only the argument of the first raising operator changes to λ, which classifies this as an unwanted term. Pulling the T_a^a through will yield an additional string of R-matrices. However, the resulting expression can be simplified using (3.55). After some rearrangement we find

$$\begin{aligned}&(T_a^a(\lambda)\,|\Psi\rangle)_{\text{uw.(k)}}\\ &=\frac{-\lambda_k^L}{\lambda-\lambda_k}\left(\prod_{j\neq k}^{M}\frac{1}{\lambda_k-\lambda_j}\right)R^{c_{k-1}a_{k+1}}_{c_{k+1}b_{k+1}}(\lambda_k-\lambda_{k+1})\cdots R^{c_{M-2}a_{M-1}}_{c_{M-1}b_{M-1}}(\lambda_k-\lambda_{M-1})R^{c_{M-1}a_M}_{a\quad b_M}(\lambda_k-\lambda_M)\\ &\times T_1^a(\lambda)T_1^{a_1}(\lambda_1)\cdots T_1^{a_{k-1}}(\lambda_{k-1})T_1^{b_{k+1}}(\lambda_{k+1})\cdots T_1^{b_M}(\lambda_M)\,|\Psi\rangle\,.\end{aligned} \tag{3.64}$$

The sum of (3.63) and (3.64) is supposed to vanish, so that these terms do not contribute when acting with the transfer matrix. To simplify a bit further we can multiply by the inverse of the string of R-matrices appearing (3.63). Since it is an inverse, this can not annihilate the sum of (3.63) and (3.64), which is important since we are trying to set this sum equal to zero. If we act with an operator from the left on the sum and set the whole expression equal to zero, we can a priori only determine that the unwanted terms are in the kernel of the operator, but since inverses are invertible, the kernel is trivial. The sought after inverse can be seen, using again the relation given in (3.55), to be

$$S^{-1}=R^{d_{k-1}e_{k-1}}_{a_k\quad a_{k-1}}(\lambda_k-\lambda_{k-1})R^{d_{k-2}e_{k-2}}_{d_{k-1}a_{k-2}}(\lambda_k-\lambda_{k-2})\cdots R^{d_2e_2}_{d_3a_2}(\lambda_k-\lambda_2)R^{e_ke_1}_{d_2a_1}(\lambda_k-\lambda_1). \tag{3.65}$$

We also notice that the R-matrix at zero becomes just a permutation yielding yet another identity for the R-matrices

$$R^{a_2c_1}_{a_1b_1}(u)R^{a_3c_2}_{a_2b_2}(0)R^{a_4c_3}_{a_3b_3}(v)=R^{c_2c_1}_{a_1b_1}(u)R^{a_4c_3}_{b_2b_3}(v). \tag{3.66}$$

Now adding the unwanted terms together and multiplying by S^{-1} we obtain after some

rearrangements

$$
\begin{aligned}
&(S^{-1}(T_1^1 + T_2^2 + T_3^3)(\lambda)\,|\Psi\rangle)_{\text{uw.(k)}} \\
&= \frac{1}{\lambda - \lambda_k}\left(\prod_{j\neq k}^{M}\frac{-1}{\lambda_k - \lambda_j}\right) T_1^{e_k}(\lambda)T_1^{e_1}(\lambda_1)\cdots T_1^{a_M}(\lambda_M)\,|\Omega\rangle \\
&+ \frac{-\lambda_k^L}{\lambda - \lambda_k}\left(\prod_{j\neq k}^{M}\frac{1}{\lambda_k - \lambda_j}\right) \mathcal{T}^{\{e_{j\leq k},a_{j>k}\}}_{\{a_{j<k},e,d_{j>k}\}}(\lambda_k,\{\lambda_j\})T_1^{e}(\lambda)T_1^{a_1}(\lambda_1)\cdots T_1^{d_M}(\lambda_M)\,|\Omega\rangle\,.
\end{aligned}
\tag{3.67}
$$

We see that, as before, the problem of diagonalising the two-scalar transfer-matrix appears. Putting everything together, when we act with the transfer matrix on a linear combination of T_1^as that diagonalise the underlying two-scalar model we can replace $\mathcal{T}$ in (3.58) and (3.67) by its eigenvalue Λ, given in (3.61), and cancel the unwanted terms as well as calculate the eigenvalue of the transfer matrix. This leads to the following set of Bethe equations

$$(-1)^{K-1} = \prod_{j=1}^{M}(r_k - \lambda_j) \tag{3.68}$$

$$\lambda_k^L = (-1)^{M-K-1}\prod_{j=1}^{K}(\lambda_k - r_j) \tag{3.69}$$

$$E = \sum_{k=1}^{M}\frac{-2}{\lambda_k}, \tag{3.70}$$

where as stated before we have $(L-M)$ ϕ_1, $(M-K)$ $\phi_3^\dagger$ and K ϕ_2 and E is the eigenvalue of the Hamiltonian. Both ϕ_2 and $\phi_3^\dagger$ are right movers in a vacuum of ϕ_1. Furthermore, ϕ_2 are also right movers in a vacuum of $\phi_3^\dagger$, so the chirality is the same on the two different levels of nesting, no left-movers appear. This is why the strange convention of having $\phi_3^\dagger$ as the first type of excitations and to choose it as the vacuum of the rank two spin-chain allows for the application of the nested Bethe ansatz.

We can, however, also transform the Bethe equations, such that we have ϕ_2 as the vacuum at the second level of nesting [86, 87]. This transformation of the Bethe roots is known as dualisation and happens solely at the level of the equations. In the context of statistical physics, this dualisation is most easily understood as a type of particle-hole transformation. Historically, dualisation appeared in the Bethe equations of supersymmetric models, where different gradings of the symmetry algebra exist. However, the important attribute of the Bethe equations, which allows for dualisation is the absence of self-scattering. Since there is no scattering term involving only the auxiliary roots, we can dualise these roots, as described in the following paragraph.

We start with the Bethe equation for the auxiliary roots

$$\prod_{j=1}^{K_4}(\alpha_{3,k} - \alpha_{4,j}) = (-1)^{K_3-1}\ , \tag{3.71}$$

where we renamed the momentum-carrying roots to α_4 and the auxiliary roots to α_3 in order to match the conventions we use later. We notice that this is a polynomial equation. Put differently, we are looking for the zeros of the polynomial given by

$$P(\alpha_3) = \prod_{j=1}^{K_4}(\alpha_3 - \alpha_{4,j}) + (-1)^{K_3} = \prod_{j=1}^{K_3}(\alpha_3 - \alpha_{3,j})\prod_{j=1}^{K_4-K_3}(\alpha_3 - \tilde{\alpha}_{3-j})\,. \tag{3.72}$$

Here, in the first step we just rewrote (3.71), while in the second step we wrote the polynomial in terms of its roots. Notice that the roots $\tilde{\alpha}_{3,k}$, being zeros of the polynomial, fulfil the same Bethe equations (3.71) as the auxiliary roots. Evaluating both forms of this polynomial at one of the momentum-carrying roots $\alpha_{4,k}$ yields a relation between two scattering terms

$$P(\alpha_{4,k}) = (-1)^{K_3} = \prod_{j=1}^{K_3}(\alpha_{4,k} - \alpha_{3,j}) \prod_{j=1}^{K_4-K_3}(\alpha_{4,k} - \tilde{\alpha}_{3-j}) \tag{3.73}$$

$$\implies \prod_{j=1}^{K_3}(\alpha_{4,k} - \alpha_{3,j}) = (-1)^{K_3} \prod_{j=1}^{K_4-K_3} \frac{1}{(\alpha_{4,k} - \tilde{\alpha}_{3,j})}\,. \tag{3.74}$$

We can now plug this into the equation for the momentum carrying roots and obtain

$$\alpha_{4,k}^L = (-1)^{K_4-1} \prod_{j=1}^{K_4-K_3} \frac{1}{\alpha_{4,k} - \tilde{\alpha}_{3,j}}\,. \tag{3.75}$$

The above dualised version of the Bethe equations is given in our paper [1]. We drop the tilde over the new α_3 and relabel $K_4 - K_3$ as K_3.

The Hamiltonian exhibits some very interesting behaviour in this sector, which we will discuss in chapter 4, e.g., it is not diagonalisable anymore - eigenstates of the Hamiltonian do not span the full Hilbert space. For now, we will see what happens, if instead of a $\phi_3^\dagger$ we added the more intuitive ϕ_3.

3.3.2 The Bad

We want to describe the spin chain containing three out of the six different scalars. As mentioned before, in principle, this gives us $\binom{6}{3}$, i.e., 20 different possibilities. In the untwisted mother theory $\mathcal{N} = 4$ these possibilities are largely related by symmetries, and many of them are identical. However, twisting the model breaks most of the symmetries and hence it can make a huge difference which scalars we choose. The most natural choice seems to be ϕ_1, ϕ_2 and ϕ_3. In this section, we consider these three scalars. Interestingly, this is the sector [54] are claiming to consider. However, for the double scaling conventions used, where all $q \to \infty$, we claim they are actually considering the sector from the previous section. We appear to be the first ones who have noticed [1] that there are serious issues with diagonalising this sector.

We can attempt to use the ABA as described in the last section to try and diagonalise the Hamiltonian. The problem is that the R-matrix is scarcely filled - most entries are zero. In particular, from the RTT relation, we can not obtain an equation like (3.51). Depending on what we choose as our vacuum and first type of excitation, equation (3.51) will only work for either T_1^2 or T_1^3, but not for both. For the other we will obtain an equation

$$T_1^1(u)T_1^a(v) = T_1^1(v)T_1^a(u). \tag{3.76}$$

T_1^1 stands on the left of the T_1^a on both sides of this equation. Thus it does not allow us to pull the T_1^1 through and act on the vacuum as is necessary for the ABA. The problem is that no matter how we choose the vacuum, one type of excitation is only left moving and the other one is only right moving, but we need to pick a convention, where they move in the same direction. Since this is not possible, the Bethe ansatz fails. Does that mean we have to give up to diagonalise the Hamiltonian in this subsector? Let us first try to visualise the spin chain.

For this, we choose ϕ_1 as the vacuum, and ϕ_2 and ϕ_3 as the excitations. In our conventions the ϕ_2 are right movers, i.e., acting with the Hamiltonian will push the ϕ_2 to the right. Similary, the ϕ_3 will get pushed to the left. If we push the different types of excitations in different directions, naturally they will meet, and the question is what happens. There are in principle three options:

- a) They reflect off each other. This can not happen, because after the collision, the excitations would travel in the opposite direction with regards to before the collision. Since the excitations can only travel in one direction, this is impossible.
- b) They fly through each other. The explicit form of the Hamiltonian does not allow for that. The Hamiltonian can only exchange a ϕ_2 and a ϕ_3 if the ϕ_2 is to the right of the ϕ_3, but the ϕ_2 is coming from the left. We say the ϕ_2 and ϕ_3 represent impenetrable walls to each other in a vacuum of ϕ_1. Technically, these walls are only impenetrable in one direction, but it is the direction from which the excitations approach each other. Hence this option is also impossible.
- This only leaves us with c): the Hamiltonian can not act on the ϕ_2 and the ϕ_3, it may only change the rest of the chain.

This leads to the following result. The Hamiltonian pushes all the ϕ_2 and ϕ_3 together until they all collide and then it must annihilate the spin chain state. We consider finite chains, and hence this implies that the Hamiltonian is nilpotent, i.e.,

$$H^N = 0\,, \tag{3.77}$$

for some finite N. We call a set of fields, where the above argument holds, eclectic field content[7] and the corresponding chain an eclectic spin-chain. We use the word eclectic to stress that we have added too many different types of excitations, some of which represent impenetrable walls to some others.

Let us be explicit about what we mean by eclectic field content. The definition of eclectic field content does only depend on the flavour of the fields under consideration, i.e., it does not depend on whether the fields are bosons or fermions, or whether there are derivatives acting on the fields. We speak of eclectic field content, if either fields of all three flavours $\{1, 2, 3\}$ are present, or if fields of all three conjugates flavours $\{\bar{1}, \bar{2}, \bar{3}\}$ are present, or if fields and conjugate fields of the same flavour $\{a, \bar{a}\}$ are present. We do not consider fields, which have decoupled during the strong twisting of $\mathcal{N} = 4$, but it would be natural to count any operator, including them as having eclectic field content.

For eclectic spin chains, we know that the Hamiltonian is nilpotent. A proof for this can be found in Appendix B. It would be interesting to see whether there is some deeper reason behind the notion of eclecticism. A nilpotent Hamiltonian means, the generalised eigenvalues are all zero. While this is already interesting the explicit Jordan block structure, i.e., how many Jordan blocks of what sizes constitute the Hamiltonian remains elusive. Except for some brute force numerical examples, we were so far unable to construct a method to find this Jordan block structure.

3.4 Differentiating in the Fishnet

In the last section, we have described how to find the, possibly generalised, eigenvalues in the various three-scalar sectors. The good version only exists in the β or γ twisted models,

[7] We actually define eclectic field content to contain additional cases, where a slight alteration of the above argument holds. For the explicit definition see the next paragraph and for further arguments the proof in appendix B.

when at least two ξ_i are non-zero. The strongly-twisted model, which has captured the most attention in the literature, is the fishnet model that has only one non-zero ξ. Therefore, in this model, all three-scalar sectors are eclectic, and a Bethe ansatz will consequently not work. The fishnet theory also does not contain any fermions. Thus, the only extension to the two-scalar model, which is non-trivial on the level of the generalised eigenvalues, includes derivatives. In this section, we will look at a sector, which contains ϕ_1, ϕ_2 and one type of derivative $\partial_{1\dot{1}}$, where we labelled the derivative by two spinor indices $\alpha\dot{\alpha}$ instead of the vector index μ. We will suppress these indices for the rest of this section and instead will just write ∂.

In order to find the eigenvalues of the Hamiltonian in this sector, we will use a coordinate Bethe ansatz. The basic version of the latter was introduced in section 3.2.1. However, we now have two different types of excitations, which as for the ABA, complicates matters and usually requires a form of nesting procedure. As we will see, we are lucky in this case, and by a smart choice of conventions, we can eliminate the nesting structure. Some literature on nested CBAs can be found in [88, 20].

We start by picking a vacuum, which we choose to be ϕ_1. Consequently, ϕ_2 and ∂ are excitations travelling on this vacuum. As opposed to all other excitations we considered so far, we can have arbitrarily many derivatives at one spin chain site. Moreover, the excitations ϕ_2 and ∂ can also overlap on the same site. We will have to keep this in mind when we calculate the S-matrix between the excitations. As before, we start by assigning a momentum to each of the excitations. The energy E is then determined as a function of these momenta in the so-called dispersion relation

$$E = \sum_{j=1}^{K_\phi} \frac{-2}{e^{ip_{\phi,j}}}\,, \tag{3.78}$$

where K_ϕ is the number of ϕ_2 excitations and $p_{\phi,j}$ their corresponding lattice momenta. We obtain (3.78) by acting on a state, where the excitations are well separated.

Notice that, since the Hamiltonian cannot move a ∂ excitation without a ϕ_2 as a "catalyst", their momentum does not appear in the dispersion law. This implies there can be no momentum exchange, between a ϕ_2 and ∂ excitation during scattering, since the energy of a given eigenstate is a fixed value. As a consequence, the S-matrix between two excitations is just a scalar function, which we can determine by acting as before on two- or three-excitation states. Acting with the Hamiltonian on the state (3.13) we already found before

$$S_{\phi\phi}(p_1, p_2) = -1\,, \tag{3.79}$$

where now we have included a subscript on the S-matrix to indicate what flavour the scattered excitations carry. Similiarly we can act with the Hamiltonian on the following state

$$\begin{aligned}|\Psi\rangle = &\sum_{l=1}^{L}\sum_{k=1}^{l-1}\left(\exp(i(kp_1+lp_2)\,|\phi(k),\partial(l)\rangle + S_{\phi,\partial}(p_1,p_2)\exp(i(lp_1+kp_2)\,|\partial(k)\phi(l)\rangle\right)\\ &+\sum_{l=1}^{L} C(p_1,p_2)\exp(il(p_1+p_2))\,|\partial(l)\phi(l)\rangle\,,\end{aligned} \tag{3.80}$$

which gives us the S-matrix element

$$S_{\phi,\partial}(p_1,p_2) = \frac{e^{ip_2}}{2-e^{-ip_2}}\,. \tag{3.81}$$

$S_{\partial,\phi}(p_1,p_2)$ can be determined the same way, but it also has to be the inverse of (3.81). The last S-matrix element $S_{\partial,\partial}(p_1,p_2)$ is more difficult to determine. The reason is that since the derivatives cannot move by themselves, we cannot simply act on a two-excitation state, but need a third excitation, a ϕ_2, as a catalyst for the scattering. We obtain

$$S_{\partial,\partial}(p_1,p_2) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}\,, \tag{3.82}$$

for the last S-matrix element. Remarkably, the scattering between derivatives is governed by the same S-matrix as in the untwisted model [89]. Determining these S-matrices is the only real calculation required by the Bethe ansatz. We now proceed in the standard way of making the ansatz, which we first write down and explain afterwards

$$|\Psi\rangle = \sum_{n_1<n_2<\ldots<n_M} \sum_{\sigma} S_\sigma(\{p_i\}) e^{i\sum_j p_{\sigma(j)} n_j} \, |n_1,\ldots,n_M; f_{\sigma(1)},\ldots,f_{\sigma(M)}\rangle + \text{local terms}. \tag{3.83}$$

We sum here over permutations $\sigma \in S_M$ of M objects to include every different assignment of the M momenta to the M positions. The kets appearing in the above equation have excitations of type $f_{\sigma(i)}$ in position i. The S_σ is the product of two-particle S-matrices, each of which corresponds to a transposition and the product of the transpositions has to be the permutation σ. In general, a permutation has different decompositions into transpositions, but the Yang-Baxter equation guarantees that all of these yield the same S_σ. Usually this S_σ acts as a linear operator on the flavour of the excitations. This is equivalent to what we have seen in the case of the nested ABA. The linear operator S_σ creates the second spin-chain nested in the one we are trying to diagonalise. However, as discussed before, no ϕ and ∂ can exchange momentum so by attaching the flavour to the momentum we already diagonalised S_σ, and it is indeed only a scalar function here. We sum over all positions n_i of the excitations with the typical plane wave prefactor. Finally, we also have some terms where the excitations are at the same spin chain site, represented here by the local terms.

We still need to properly impose the periodic boundary conditions on (3.83), which leads to the Bethe equations

$$1 = e^{ip_i L} \prod_{i\neq j} S_{f_i f_j}(p_i,p_j). \tag{3.84}$$

These equations can, of course, be written in terms of the different momenta. Adding the dispersion formula and the zero-momentum condition, which takes care of the identification of cyclically permuted states, we summarise the result of this section by the following four equations

$$E = \sum_{j=1}^{K_R} \frac{-2}{e^{ip_{\phi,j}}}\,, \tag{3.85}$$

$$e^{ip_{\phi,k}L} = (-1)^{K_R-1} \prod_{j=1}^{K_\partial} \frac{2 - e^{-ip_{\partial,j}}}{e^{ip_{\partial,j}}}\,, \tag{3.86}$$

$$e^{ip_{\partial,k}L} = \left(\frac{e^{ip_{\partial,k}}}{2 - e^{-ip_{\partial,k}}}\right)^{K_R} \prod_{j\neq k}^{K_\partial} -\frac{e^{i(p_{\partial,k}+p_{\partial,j})} - 2e^{ip_{\partial,j}} + 1}{e^{i(p_{\partial,j}+p_{\partial,k})} - 2e^{ip_{\partial,k}} + 1}\,, \tag{3.87}$$

$$1 = \prod_{j=1}^{K_R} e^{ip_{\phi,j}} \prod_{j=1}^{K_\partial} e^{ip_{\partial,j}}\,. \tag{3.88}$$

As for all sectors, we will discuss the content of these equations in the next chapter. In principle, one can now try to add the other types of derivatives to this sector. Alternatively, one could attempt to add derivatives or fermions in the β-twisted model. One may choose to see whether either the CBA or ABA can work in these extended sectors and the principle ideas should be clear. However, we have now quite a few Bethe equations, for which we can say with confidence that they describe the discussed sectors. Thus, let us see whether we can find a procedure to scale the twisted but unscaled Bethe equations, which would yield a shortcut to the Bethe equations. This is the topic of the next section.

3.5 Scaling the Beauty and the Twist Equations

In this final section of the current chapter on methods, we find a way to scale the twisted Bethe equations found by Beisert and Roiban in [47]. The equations can be written in several different conventions, mainly due to the option to pick so-called gradings. These gradings correspond loosely to different ways to pick new vacuua at each level of nesting. We will use only two different gradings, for which we have summarised the equations in Appendix A.

The problem presented to us is the following. The unscaled equations contain factors of q, which we want to send to infinity. The unscaled equations also contain many variables, the Bethe roots u_j. For the equations to be correct these variables need to scale with q, however, there can a priori be many different options for scaling the set of Bethe roots, which can provide the required factors of q. Our task is to determine which general scaling works for the solutions of these equations. Hereby we rely on the insights won from doing the Bethe ansatz and the equations for specific sectors, which should be reproduced by our scaling procedure.

There are several different places one could start. Technically, the easiest way is to consider the zero-momentum condition

$$1 = q^{2(K_R-K_L)} \prod_{j=1}^{K_R+K_L+K_{\rm nd}} \frac{u_{4,j}+i/2}{u_{4,j}-i/2}\,, \tag{3.89}$$

where K_R, K_L and $K_{\rm nd}$ are the number of right-movers, left-movers and non-dispersing excitations[8] respectively. Scaling the $u_{4,j}$ as a positive power of q will not help in cancelling the factor in front of the product. To cancel this factor the roots need to go to $\pm i/2$, with a subleading term, which has to be some negative power of q. The zero-momentum condition seems to suggest the simplest scaling of the momentum-carrying Bethe roots to be

$$u_{4,j} \to \begin{cases} -i/2 - iq^{-2}\alpha_{4,j}\,, & j = 1,\dots,K_R\,, \\ +i/2 + iq^{-2}\alpha'_{4,j-K_R}\,, & j = K_R+1,\dots,K_R+K_L\,, \\ \tilde{u}_{4,j-K_R-K_L}\,, & j = K_R+K_L+1,\dots,K_4\,, \end{cases} \tag{3.90}$$

where $K_4 = K_R + K_L + K_{\rm nd}$ is just the total number of excitations, and the Bethe roots of the strongly-twisted model $\alpha_{4,j}$, $\alpha'_{4,j-K_R}$ and $\tilde{u}_{4,j-K_R-K_L}$ are independent of q. We can substitute this scaling into the dispersion relation and find

$$E = \sum_{j=1}^{K_R} \frac{-2}{\alpha_{4,j}} + \sum_{j=1}^{K_L} \frac{-2}{\alpha'_{4,j}}\,. \tag{3.91}$$

[8] Non-dispersing excitations are non-movers and derivatives.

Notice that, since we are talking about one-loop results, we have an implicit coupling constant in front of the energy. To change the coupling constant from the untwisted g^2 to ξ^2, we have an additional factor of q^2, which we have already removed from equation (3.91). So far we have not considered left-movers, but in principle, they should contribute to the energy the same way as the right movers, and indeed we see that they do. Furthermore, it is reassuring that the dispersion relation (3.91) agrees with all the sectors for which we have derived Bethe equations. The roots corresponding to non-dispersing excitations do not enter the dispersion relations, as expected. Finally, the zero-momentum condition using the scaling from (3.90) becomes

$$\prod_{j=1}^{K_R} \alpha_{4,j} \prod_{j=1}^{K_L} \frac{1}{\alpha'_{4,j}} \prod_{j=1}^{K_{\mathrm{nd}}} \frac{\tilde{u}_{4,j}+i/2}{\tilde{u}_{4,j}-i/2} = 1 \tag{3.92}$$

in the large q limit.

In general, when performing a Bethe ansatz, we sum over all different assignments of the momenta to the excitations. This would mean that there are not two different set of momenta for right- and left-moving excitations, but that all momenta get paired with all types of excitations in principle. However, as we have seen in the explicit example of the CBA done in section 3.4, the S-matrix is transmission diagonal. This means that unlike the usual case, the momenta are tied to the three types of excitations: right-moving, left-moving and non-dispersing. Hence it seems reasonable that this separation of momentum carrying roots appears for the strongly-twisted models in all sectors. We will now argue how to scale the auxiliary roots for different sectors. Since this differs from sector to sector, we will treat them separately.

3.5.1 Differentiating the Fishnet Again

Let us first look at the fishnet model, including the remaining derivatives; this then gives the complete non-eclectic part of the fishnet model. We have operators constructed out of ϕ_1 and ϕ_2, with an arbitrary number of all the derivatives $\partial_{1,\dot{1}}$, $\partial_{1,\dot{2}}$, $\partial_{2,\dot{1}}$ and $\partial_{2,\dot{2}}$ acting on the fields. The unscaled twisted Bethe equations are given in appendix A.2, where as usual ϕ_1 is chosen as the vacuum. We have to fix $K_3 = K_5$ and $K_1 = K_7 = 0$ to restrict the spin-chain to the field content mentioned above. Then we have $K_4 - K_5$ right-movers and K_5 non-dispersing excitations, which are in this case derivatives.

We start by scaling the momentum-carrying-roots as described in the introduction to this section. This leads to an equation for the right-moving momentum carrying roots, where the factors of q^2 already cancel. For the non-dispersing roots we require the scattering with the u_3 or u_5 roots to produce a factor of q^{2L} as seen in the following equation

$$\left(\frac{\tilde{u}_{4,k}+i/2}{\tilde{u}_{4,k}-i/2}\right)^L = q^{-2L} \prod_{j=1}^{K_3} \frac{\tilde{u}_{4,k}-u_{3,j}-i/2}{\tilde{u}_{4,k}-u_{3,j}+i/2} \prod_{j=1}^{K_5} \frac{\tilde{u}_{4,k}-u_{5,j}-i/2}{\tilde{u}_{4,k}-u_{5,j}+i/2} \times (\mathrm{q-independent\ factor})\,. \tag{3.93}$$

Here we left the scattering between the momentum-carrying roots unspecified, because in the scaling limit their q-dependence should vanish. In the equations for the auxiliary roots u_3 and u_5, the inverse scattering terms appear together with a factor q^L. This suggests, together with the fact that we have K_3 roots of type u_3, u_5 and $\tilde{u}_4$, that we should identify one root from each set. Explicitly, it seems natural to take

$$u_{3,j} = \tilde{u}_{4,j} + i/2 + iq^{-L}\beta_{3,j}\,, \qquad u_{5,j} = \tilde{u}_{4,j} + i/2 + iq^{-L}\beta_{5,j}\,, \tag{3.94}$$

while leaving the u_2 and u_6 roots unscaled. This also ties in nicely with our observation from section 3.4 that the S-matrix acting on the flavour basis is already diagonal. We hence expect some pairing of the Bethe roots, corresponding to specific types of excitations. Indeed, the above scaling removes all factors of q^2 from the equations, and furthermore, it does reproduce our results from the CBA. Finally, it is also the unique scaling, which works for single excitations. All in all, it appears that this has to be at least one solution and probably the unique scaling of the Bethe equations.

The scaled equations are

$$\alpha_{4,k}^{L} = (-1)^{K_R-1} \prod_{j=1}^{K_\partial} \frac{(\tilde{u}_{4,j}+3i/2)(\tilde{u}_{4,j}-i/2)}{(\tilde{u}_{4,j}+i/2)(\tilde{u}_{4,j}+i/2)}\,, \tag{3.95}$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_\partial} \frac{u_{2,k}-\tilde{u}_{4,j}}{u_{2,k}-\tilde{u}_{4,j}-i}\,, \tag{3.96}$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_\partial} \frac{u_{6,k}-\tilde{u}_{4,j}}{u_{6,k}-\tilde{u}_{4,j}-i}\,, \tag{3.97}$$

$$\begin{aligned}\left(\frac{\tilde{u}_{4,k}+i/2}{\tilde{u}_{4,k}-i/2}\right)^{L-K_R} &= \left(\frac{\tilde{u}_{4,k}+i/2}{\tilde{u}_{4,k}+3i/2}\right)^{K_R} \prod_{j\neq k}^{K_\partial} \frac{\tilde{u}_{4,k}-\tilde{u}_{4,j}-i}{\tilde{u}_{4,k}-\tilde{u}_{4,j}+i} \\ &\quad \times \prod_{j=1}^{K_2} \frac{\tilde{u}_{4,k}-u_{2,j}+i}{\tilde{u}_{4,k}-u_{2,j}} \prod_{j=1}^{K_6} \frac{\tilde{u}_{4,k}-u_{6,j}+i}{\tilde{u}_{4,k}-u_{6,j}}\,.\end{aligned} \tag{3.98}$$

We also know the energy and zero-momentum condition (3.91) and (3.92), with $K_R = K_4 - K_5$ and $K_{\text{nd}} = K_5 = K_\partial$ as mentioned before. Since these only depend on the momentum carrying roots they are sector independent[9].

This is the complete set of Bethe equations for the fishnet model. Next, we will look at the β-twisted theory, in which we can add a third scalar, derivatives and fermions and consequently the possible non-eclectic field content is much larger.

3.5.2 Three Scalars and Two Fermions

So far, in the β-twisted theory, we have described the two-scalar and three-scalar sectors. Given our insight how to scale the momentum carrying roots, we will now add fermions. A good point to start is the sector containing the fields ϕ_1, ϕ_2 $\phi_3^\dagger$, $\bar{\psi}_{3,\dot{1}}$ and $\bar{\psi}_{3,\dot{2}}$. The corresponding unscaled twisted Bethe equations can be found in appendix A.1 In this sector, all excitations are right-movers and hence $K_R = K_4$. This time around, there is a factor of q^{2K_3} in the equation for the momentum carrying roots, which has to come from the scattering with the u_3 roots. There is also a factor of q^{2K_4}, in the equation for auxiliary roots, which has to come from the inverse of this scattering term. Since we know from section 3.3.1 that in this case there is a scattering term between the momentum carrying and auxiliary roots and we also know that the scaled roots come with a factor of q^{-2} the most natural assumption is that each S-matrix produces a factor of q^{-2}. This can be achieved by the scaling of the u_3 roots as

$$u_{3,j} = -iq^{-2}\alpha_{3,j}\,. \tag{3.99}$$

[9] One still has to correctly identify for each sector which excitations are left-movers, right-movers and non-dispersing.

After this rescaling, all factors of q in the Bethe equations cancel and no other roots have to be scaled. Substituting (3.90) and (3.99) into the Bethe equations and taking the $q \to \infty$ limit immediately yields

$$1 = \prod_{j=1}^{K_4} \alpha_{4,j} \,, \tag{3.100}$$

$$1 = \prod_{j\neq k}^{K_1} \frac{u_{1,k} - u_{1,j} - i}{u_{1,k} - u_{1,j} + i} \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + i/2}{u_{1,k} - u_{2,j} - i/2} \,, \tag{3.101}$$

$$\left(\frac{u_{2,k} + i/2}{u_{2,k} - i/2}\right)^{K_3} = \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + i/2}{u_{2,k} - u_{1,j} - i/2} \,, \tag{3.102}$$

$$1 = (-1)^{K_3-1} \prod_{j=1}^{K_4} (\alpha_{4,k} - \alpha_{3,j}) \prod_{j=1}^{K_2} \frac{u_{2,j} + i/2}{u_{2,j} - i/2} \,, \tag{3.103}$$

$$\alpha_{4,k}^L = (-1)^{K_4-1} \prod_{j=1}^{K_3} \frac{1}{\alpha_{4,k} - \alpha_{3,j}} \,, \tag{3.104}$$

$$E = -\sum_{j=1}^{K_4} \frac{2}{\alpha_{4,j}} \,, \tag{3.105}$$

where we have $(L - K_4)$ ϕ_1, $(K_4 - K_3)$ ϕ_2, $(K_3 - K_2)$ $\phi_3^\dagger$ and K_2 $\bar{\psi}_3$. The scaling produces the correct result for the good three-scalar sector. Since we have no additional scaling beyond that sector, we take the above as the unique way to scale the Bethe roots in this sector.

3.5.3 Two Scalars, Two Fermions and Derivatives

One can also include derivatives and fermions together in the β-scaled model. To exemplify this point, we consider the sector containing the fields ϕ_1, $\psi_{1\alpha}$, $\phi_3^\dagger$ and $\bar{\psi}_{3\dot{\alpha}}$, with all four types of derivatives $\partial_{\alpha\dot{\alpha}}$. Here an additional complexity arises, since two fermions can join to form a derivative and a derivative can split up into two fermions under the action of the Hamiltonian. This means we can have a process like

$$\partial_{1\dot{1}} \phi_3^\dagger \phi_1 \to \psi_{11} \bar{\psi}_{3\dot{1}} \,.$$

The quantum numbers on the two sides of this process are identical. However, for the scaling of the Bethe ansatz, this does not make a significant difference.

The unscaled twisted Bethe equations are given in appendix A.2. Note that we use the R-symmetry rotated version of the equations, meaning the excitations are given in the right column of table A.2. Here again ϕ_1 is taken as the vacuum, and we set $K_1 = K_7 = 0$. We have $K_R = K_4 - K_5$ right-movers and K_5 non-dispersing excitations, which in this case include both derivatives and ψ_{11}. A scaling that works is simply to let

$$u_{5,j} = i/2 + \tilde{u}_{4,j} + iq^{-2L}\beta_j \,, \tag{3.106}$$

which is also in accordance with the scaling in the fishnet theory, discussed in section 3.5.1. In contrast to that section, the u_3 roots do not change the nature of the excitations from right-moving to non-dispersing, and hence we do not expect them to scale. Indeed, together with the standard scaling of the momentum-carrying roots (3.90), equation (3.106) is sufficient to cancel all the factors of q in the Bethe equations. After the substitution and

taking the limit $q \to \infty$ we find the Bethe equations in the strongly-β-twisted theory in this sector

$$\alpha_k^L = (-1)^{K_R-1} \prod_{j=1}^{K_3} \frac{u_{3,j}+i}{u_{3,j}} \prod_{j=1}^{K_5} \frac{\tilde{u}_{4,j}-i/2}{\tilde{u}_{4,j}+i/2} \tag{3.107}$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+i/2}{u_{2,k}-u_{3,j}-i/2}\,, \tag{3.108}$$

$$\left(\frac{u_{3,k}+i}{u_{3,k}}\right)^{K_R} = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+i/2}{u_{3,k}-u_{2,j}-i/2} \prod_{j=1}^{K_5} \frac{u_{3,k}-\tilde{u}_{4,j}-i/2}{u_{3,k}-\tilde{u}_{4,j}+i/2}\,, \tag{3.109}$$

$$\left(\frac{\tilde{u}_{4,k}+i/2}{\tilde{u}_{4,k}-i/2}\right)^{L-K_R} = \prod_{j=1}^{K_3} \frac{\tilde{u}_{4,k}-u_{3,j}-i/2}{\tilde{u}_{4,k}-u_{3,j}+i/2} \prod_{j=1}^{K_6} \frac{\tilde{u}_{4,k}-u_{6,j}+i}{\tilde{u}_{4,k}-u_{6,j}}\,, \tag{3.110}$$

$$1 = \prod_{j=1}^{K_5} \frac{u_{6,k}-\tilde{u}_{4,j}}{u_{6,k}-\tilde{u}_{4,j}-i} \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i}\,. \tag{3.111}$$

3.5.4 The Largest Sector

Let us see whether we can extend the scaling of the roots to even larger sectors. We would like to have scaled Bethe equations for the largest possible non-eclectic field content. There are a lot of equivalent sectors, which have maximal non-eclectic field content, and for some of these, we have more insight regarding the scaling of the Bethe roots than for others. In particular, we should try to choose a vacuum and excitations such that all excitations are right-movers. Choosing the standard vacuum ϕ_1 defines the field content as $\{\phi_1, \psi_1, \phi_2, \psi_2, \phi_3^\dagger, \bar{\psi}_3, \partial\}$, where we suppressed the space-time indices. We use the equations from appendix A.2 in the non-rotated form and in order to avoid eclectic field content we set $K_7 = 0$ and $K_3 \geq K_5$.

As stated in (3.90) we know to scale the momentum-carrying roots as:

$$u_{4,j} = \frac{-i}{2} - i\alpha_{4,j} q^{-2} \qquad \text{for } j = 1, \ldots, K_4 - K_5\,, \tag{3.112}$$

$$u_{4,j} = \tilde{u}_{4,j-K_4+K_5} \qquad \text{for } j = K_4 - K_5 + 1, \ldots, K_4\,. \tag{3.113}$$

As usual the question of how to scale the auxilliary roots is more difficult. From the equations for the second set of momentum carrying roots, we see that the pairing of u_5 roots with the $\tilde{u}_4$ roots as given by

$$u_{5,j} = \frac{i}{2} + \tilde{u}_{4,j} + i q^{-2(L-K_4+K_3)} \beta_{5,j} \tag{3.114}$$

cancels the factors of q in said equations. It follows straightforwardly that the u_6 roots do not have to be scaled. We have seen before that the Bethe roots appear to be attached to specific flavours. Since we assign the u_3 roots to $K_3 - K_5$ fields of one flavour and K_5 fields of another, it is natural to scale K_5 of the u_3 roots differently from the rest. Together with the scaling from the smaller sectors we are lead to the scaling

$$u_{3,j} = -i - i\alpha_{3,j} q^{-2} \qquad \text{for } j = 1, \ldots, K_3 - K_5\,, \tag{3.115}$$

$$u_{3,j} = \frac{i}{2} + \tilde{u}_{4,j-K_3+K_5} + i\beta_{3,j} q^{-2(K_4-K_5)} \qquad \text{for } j = K_3 - K_5 + 1, \ldots, K_3\,, \tag{3.116}$$

for the remaining roots. We do not have to scale the roots u_1 or u_2, which appears appropriate since they do not change the flavour of the excitations.

Plugging this set of roots into the twisted, but unscaled, Bethe equations see Appendix A.2, we obtain the Bethe equations for the largest non-eclectic sector of the strongly-β-twisted theory. The $\beta_{3,k}$ and $\beta_{5,k}$ roots can be eliminated and we are left with the following equations

$$\alpha_{4,k}^{L} = (-1)^{K_4-K_5-1} \prod_{j=1}^{K_3-K_5} (\alpha_{3,j} - \alpha_{4,k}) \prod_{j=1}^{K_5} \frac{(\tilde{u}_{4,j} - i/2)(\tilde{u}_{4,j} + 3i/2)}{(\tilde{u}_{4,j} + i/2)^2}\,, \tag{3.117}$$

$$1 = \prod_{j=1}^{K_4-K_5} \frac{1}{(\alpha_{3,k} - \alpha_{4,j})} \prod_{j=1}^{K_2} \frac{\tilde{u}_{2,j} + i/2}{\tilde{u}_{2,j} + 3i/2} \prod_{j=1}^{K_5} \frac{\tilde{u}_{4,j} + 3i/2}{\tilde{u}_{4,j} + i/2}\,, \tag{3.118}$$

$$1 = \prod_{j=1}^{K_2} \frac{\tilde{u}_{1,k} - \tilde{u}_{2,j} + i/2}{\tilde{u}_{1,k} - \tilde{u}_{2,j} - i/2}\,, \tag{3.119}$$

$$1 = \prod_{j=1}^{K_1} \frac{\tilde{u}_{2,k} - \tilde{u}_{1,j} + i/2}{\tilde{u}_{2,k} - \tilde{u}_{1,j} - i/2} \prod_{j\neq k}^{K_2} \frac{\tilde{u}_{2,k} - \tilde{u}_{2,j} - i}{\tilde{u}_{2,k} - \tilde{u}_{2,j} + i} \prod_{j=1}^{K_5} \frac{\tilde{u}_{2,k} - \tilde{u}_{4,j}}{\tilde{u}_{2,k} - \tilde{u}_{4,j} - i}\,, \tag{3.120}$$

$$\begin{aligned} 1 = {} & \left(\frac{\tilde{u}_{4,k} - i/2}{\tilde{u}_{4,k} + i/2}\right)^{L-K_4+K_5} \left(\frac{\tilde{u}_{4,k} + i/2}{\tilde{u}_{4,k} + 3i/2}\right)^{K_4+K_3-2K_5} \prod_{j\neq k}^{K_5} \frac{\tilde{u}_{4,k} - \tilde{u}_{4,j} - i}{\tilde{u}_{4,k} - \tilde{u}_{4,j} + i} \\ & \times \prod_{j=1}^{K_2} \frac{\tilde{u}_{4,k} - \tilde{u}_{2,j} + i}{\tilde{u}_{4,k} - \tilde{u}_{2,j}} \prod_{j=1}^{K_6} \frac{\tilde{u}_{4,k} - \tilde{u}_{6,j} + i}{\tilde{u}_{4,k} - \tilde{u}_{6,j}}\,, \end{aligned} \tag{3.121}$$

$$1 = \prod_{j=1}^{K_5} \frac{\tilde{u}_{6,k} - \tilde{u}_{4,j}}{\tilde{u}_{6,k} - \tilde{u}_{4,j} - i} \prod_{j\neq k}^{K_6} \frac{\tilde{u}_{6,k} - \tilde{u}_{6,j} - i}{\tilde{u}_{6,k} - \tilde{u}_{6,j} + i}\,. \tag{3.122}$$

These are our conjectured Bethe equations for the maximal non-eclectic sector of the β-twisted theory. When discussing our results in chapter 4, we will also give additional arguments to support these equations.

Before closing this chapter, we want to state that the scaling in terms of inverse powers of q for the general strongly-γ-deformed theory is the same as for the strongly-β-twisted theories. Nevertheless, some additional factors appear corresponding to the relative size of the coupling constants. If we let $\xi_i = a_i \xi$ with respect to the reference coupling ξ, the Bethe equations for the maximal non-eclectic sector are

$$\begin{aligned} \alpha_{4,k}^{L} = {} & a_1^{-K_1-K_3+K_5} a_2^{-K_1-K_3+K_5} a_3^{-2L+2K_1} (-1)^{K_4-K_5-1} \\ & \times \prod_{j=1}^{K_3-K_5} (\alpha_{3,j} - \alpha_{4,k}) \prod_{j=1}^{K_5} \frac{(\tilde{u}_{4,j} - i/2)(\tilde{u}_{4,j} + 3i/2)}{(\tilde{u}_{4,j} + i/2)^2}\,, \end{aligned} \tag{3.123}$$

$$1 = a_1^{K_4-K_5} a_2^{-L+K_1} a_3^{L-K_1-K_4+K_5} \prod_{j=1}^{K_4-K_5} \frac{1}{(\alpha_{3,k} - \alpha_{4,j})} \prod_{j=1}^{K_2} \frac{\tilde{u}_{2,j} + i/2}{\tilde{u}_{2,j} + 3i/2} \prod_{j=1}^{K_5} \frac{\tilde{u}_{4,j} + 3i/2}{\tilde{u}_{4,j} + i/2}\,, \tag{3.124}$$

$$1 = a_1^{K_4-K_5} a_2^{-L-K_3-K_4} a_3^{L+K_3+K_4+K_5} \prod_{j=1}^{K_2} \frac{\tilde{u}_{1,k} - \tilde{u}_{2,j} + i/2}{\tilde{u}_{1,k} - \tilde{u}_{2,j} - i/2}\,, \tag{3.125}$$

$$1 = \prod_{j=1}^{K_1} \frac{\tilde{u}_{2,k} - \tilde{u}_{1,j} + i/2}{\tilde{u}_{2,k} - \tilde{u}_{1,j} - i/2} \prod_{j\neq k}^{K_2} \frac{\tilde{u}_{2,k} - \tilde{u}_{2,j} - i}{\tilde{u}_{2,k} - \tilde{u}_{2,j} + i} \prod_{j=1}^{K_5} \frac{\tilde{u}_{2,k} - \tilde{u}_{4,j}}{\tilde{u}_{2,k} - \tilde{u}_{4,j} - i}\,, \tag{3.126}$$

$$\begin{aligned} 1 = {} & a_1^{K_4-K_5} a_2^{L-K_1-K_4+K_5} a_3^{-L+K_1} \left(\frac{\tilde{u}_{4,k} - i/2}{\tilde{u}_{4,k} + i/2}\right)^{L-K_4+K_5} \left(\frac{\tilde{u}_{4,k} + i/2}{\tilde{u}_{4,k} + 3i/2}\right)^{K_4+K_3-2K_5} \\ & \times \prod_{j\neq k}^{K_5} \frac{\tilde{u}_{4,k} - \tilde{u}_{4,j} - i}{\tilde{u}_{4,k} - \tilde{u}_{4,j} + i} \prod_{j=1}^{K_2} \frac{\tilde{u}_{4,k} - \tilde{u}_{2,j} + i}{\tilde{u}_{4,k} - \tilde{u}_{2,j}} \prod_{j=1}^{K_6} \frac{\tilde{u}_{4,k} - \tilde{u}_{6,j} + i}{\tilde{u}_{4,k} - \tilde{u}_{6,j}}\,, \end{aligned} \tag{3.127}$$

$$1 = \prod_{j=1}^{K_5} \frac{\tilde{u}_{6,k} - \tilde{u}_{4,j}}{\tilde{u}_{6,k} - \tilde{u}_{4,j} - i} \prod_{j\neq k}^{K_6} \frac{\tilde{u}_{6,k} - \tilde{u}_{6,j} - i}{\tilde{u}_{6,k} - \tilde{u}_{6,j} + i}\,. \tag{3.128}$$

To conclude this chapter, we briefly recall the main achievements in the methodological approach. Firstly, we derived the dilatation operator of strongly-twisted $\mathcal{N} = 4$ SYM. Starting from first principles, we then derive in a second step a Bethe ansatz for these theories. Finally, we provide a shortcut to derive the Bethe equations directly from those of the unscaled twisted models. Given these tools, we will proceed in the next chapter to analyse the spectrum.

Chapter 4

Spectrum: Analysis

In the preceding chapter we have explained several different methods to diagonalise the Hamiltonian of the strongly-twisted models. This has led to several sets of Bethe equations describing different sectors. We will now analyse these equations as well as their solutions. As before, we work our way up from the simplest - the two-scalar sector - to the maximal non-eclectic sector. Furthermore, by comparing the results from the Bethe equations to some low-length direct diagonalisation, we will discuss several shortcomings of the Bethe ansatz in these models. Most notably, as mentioned before the Hamiltonians we investigate are not hermitian, and some of them are not diagonalisable. The Bethe ansatz will not find the Jordan block structure of these Hamiltonians but is limited to only find eigenstates.

4.1 The Two-Scalar Sector

Let us start with the simplest possible sector, where we have only two scalars ϕ_1 and ϕ_2. We calculated the corresponding Bethe equations in three different ways in section 3.2. They are

$$E = -2\sum_{k=1}^{M}(e^{-ip_k})\,, \tag{4.1}$$

$$1 = \prod_{k=1}^{M} e^{-ip_k}\,, \tag{4.2}$$

$$e^{-ip_k L} = (-1)^{M-1}, \tag{4.3}$$

where E is the eigenvalue of the Hamiltonian, $L-M$ and M are the number of ϕ_1 and ϕ_2 respectively and the last equation holds for all $k = 1, \ldots, M$.

The first thing we notice is that these equations provide as many eigenvalues as expected for a diagonalisable operator. When we ignore the constraint that the states are supposed to be invariant by a shift of an integer amount of spin-chain sites, we have $\binom{L}{M}$ states, corresponding to the positions of the M ϕ_2. There is also $\binom{L}{M}$ different ways to choose different Lth roots of (negative) unity. When we impose the constraint that our states are supposed to be shift invariant through the zero-momentum condition, we remove only the non-shift invariant states and no others. So the Hamiltonian is diagonalisable, and our Bethe equations yield all the eigenvalues. In principle, the eigenstates can be obtained by plugging the Bethe roots into the ansatz we made.

We chose ϕ_2 as right-movers in a vacuum of ϕ_1. Alternatively, we could have chosen ϕ_1 as left-movers in a vacuum of ϕ_2. Since this is only a convention, we need to obtain the

same eigenvalues, when we replace all ϕ_1 with ϕ_2 and vice versa. Thus, since the Bethe equations are complete, they need to yield the same solutions for the energies under the replacement $M \to L - M$. We can check this as follows. Assume that we have a certain set of solutions to the Bethe equations for M magnons, i.e., that we have chosen M roots of (negative) unity. If we take the remaining $L - M$ roots and multiply them by -1, we get a solution to the equations with $L - M$ magnons with the same energy. This is because

- Lth roots of (negative) unity add to 0. If we pick a subset of roots and add them up to E, the sum of the remaining roots will give $-E$ and hence -1 times the remaining roots adds up to E again.
- We can write the negative of the roots as $e^{-ip_k L+\pi L}$ from which we can extract the $(-1)^L$ needed to change the $(-1)^{M-1}$ to a $(-1)^{L-M-1}$ on the right hand side of the Bethe equations (4.3).
- The zero-momentum condition is the trickiest. However, notice that the product of all the possible Bethe roots is $(-1)^{L-M}$. Since the product of the set of M Bethe roots chosen is 1, the product of the remaining $L - M$ roots is $(-1)^{L-M}$. Taking the negative of all of them will cancel this factor, and the zero-momentum condition (4.2) is also fulfilled.

In conclusion, the Bethe equations yield the same energies, if we replace M by $L - M$, which is in contrast to the Bethe equations for the Heisenberg spin-chain, where we have a so-called beyond the equator problem. Of course, we only need to calculate solutions up to $M = L/2$ since, beyond that point, the energies will just repeat themselves. The eigenstates will just be mirrored versions of the ones found for $M < L/2$.

Let us now look at some specific solutions of the Bethe equations, i.e., energies. We restrict our discussion to the case $L > 2$ to avoid the prewrapping corrections mentioned in section 2.2.2. The case $M = 1$ is trivial since there is only one shift invariant eigenstate and it has eigenvalue -2. For the case $M = 2$, due to the zero-momentum condition we have to choose Bethe roots which are complex conjugates of each other, which will lead to the real eigenvalues $-4\cos(n\pi/L)$, for any integer n. The more interesting solutions start at $M = 3$. For this case, we choose three Lth roots of unity as our Bethe roots, whose product is 1. The sum of these Bethe roots - $\frac{E}{-2}$ - can be, and generically is, complex.

Of course, the sum of three complex numbers on the unit circle is in absolute value always less than three, so we can constrain the region in which the energy lies to a disc of radius six. In fact, we can do much better by only imposing the zero-momentum condition. Thus, let us ignore the actual Bethe equations for the moment since they depend on L and look at the sum of any three numbers on the unit circle, whose product is one. What is this sum, which we also call with a slight abuse of notation $\frac{E}{-2}$? If we pick one of these numbers to be $e^{i\alpha}$, the sum of the remaining numbers lies on the line $re^{-i\alpha/2}$, for $-2 \leq r \leq 2$. We can recast this statement into the following equation

$$\mathrm{Im}(E) = \sin(\alpha) + \tan\left(\frac{\alpha}{2}\right)(\mathrm{Re}(E) + \cos(\alpha)), \tag{4.4}$$

for $0 \leq \alpha \leq 2\pi/3$. We can maximise this for our choice of α to obtain an upper bound for the imaginary part of E in terms of the real part of E. It is given by

$$\mathrm{Im}(E)_{\max} = \frac{(2\sqrt{3 - \mathrm{Re}(E)} + \mathrm{Re}(E))\sqrt{3 - \sqrt{3 - \mathrm{Re}(E)}}}{\sqrt{1 + \sqrt{3 - \mathrm{Re}(E)}}}, \tag{4.5}$$

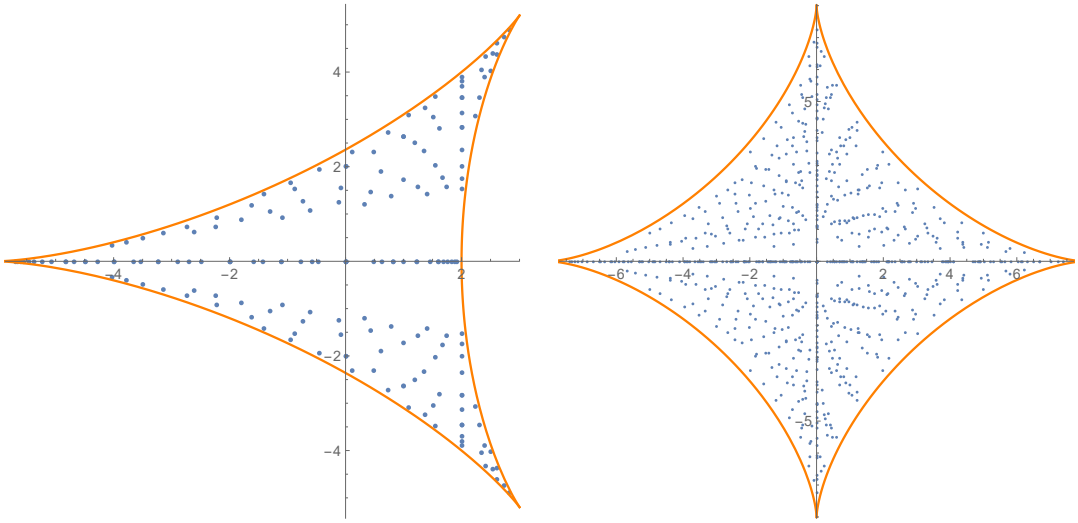

Figure 4.1: On the left, we see the energies for $M = 3$ up to $L = 16$, plotted in the complex E-plane. On the right, energies for $M = 4$ up to $L = 18$ are shown. We also marked in orange the derived bounds for the energies.

which is valid for $-6 \leq \mathrm{Re}(E) \leq 3$. The region in which E lies has to be invariant under a rotation by 120 degrees since the product of three numbers does not change if we rotate them all in the complex plane by 120 degrees. Hence the zero-momentum condition is invariant under this transformation. Rotating the above bound twice by 120 degrees and glueing the resulting curves together yields the allowed region for the energy shown in orange in figure 4.1.

In the same figure, we have plotted the actual energies of the two-scalar sector up to length 16, and as we can see, they indeed all lie within the allowed region. The energies belonging to one common root $e^{i\alpha}$ all lie on a straight line, one very nice example of which we can see for the root -1, which creates the vertical line at $\mathrm{Re}(E) = 2$. There is also a clear symmetry around the real axis, i.e., the solutions come in complex conjugate pairs, as expected for a real matrix. However, note that the rotational symmetry we had when working with arbitrary numbers on the unit circle has been broken by the Bethe equations, which allow only for discrete values of the Bethe roots. There also appears to be a gap around the real axis for larger real parts of the energy. The values on the real axis include the Bethe root 1 when displacing this root away from the real axis the sum of the other two roots gets displaced in the same direction if their real part is close to -1. Two roots, with real part close to -1 correspond to the right side of the diagram, and hence, since all three imaginary parts of the three Bethe roots corresponding to this part of the diagram have the same sign, they add up to create the gap around the real axis. However, as $L \to \infty$ this gap will close, and energies with real part equal to 2 can have arbitrarily small imaginary part. In fact, if we increase L, the entire region will be densely filled with solutions for the energy, since there will be Bethe roots arbitrarily close to any point on the unit circle. As a consequence, within arbitrary accuracy any sum of numbers on the unit circle will yield an energy and the problem will become equivalent to the one we solved to determine the allowed region for energies.[1]

[1]Technically we have not shown that all values in the allowed region are actually obtained, even if we ignore the Bethe equations and consider any three roots of unity. However, from the idea of picking one root of unity and drawing a straight line through it, with length 8 and then watching what area this line traces as we change the root of unity it is clear that indeed any point in the entire region is obtainable as a

We can do the same analysis for the case of four magnons, but we will keep it short and only state the results. The boundary in the first quadrant for four magnons is given by

$$\mathrm{Im}(E) = \left(4 - (\mathrm{Re}(E))^{\frac{2}{3}}\right)^{\frac{3}{2}}, \tag{4.6}$$

which is now valid from $0 \leq \mathrm{Re}(E) \leq 8$. This curve can now be rotated three times by 90 degrees to give the remaining parts of the boundary in which the energies have to lie. The resulting allowed region for the energies of the four magnon states are shown in figure 4.1 together with the energies up to length 18 and as we can see all the energies are within the region as predicted. The rest of our analysis for $M = 3$ basically carries over unmodified, except that the Bethe roots are now roots of negative unity.

For larger M, the allowed region will be a curved M-gon, however, if this is staying convex as we saw for small M or is becoming concave is still unclear. It also appears natural to assume that the allowed region will be densely filled for $L \to \infty$, although this also remains a conjecture. As we saw before, adding a third scalar in the fishnet model will yield eclectic field content. Consequently, when we restrict our attention to the fishnet model, all Bethe equations will have this two-scalar model as an underlying spin-chain, and we will see it as a core part several times again.

4.2 The Three-Scalar Sector

Just as the two-scalar model, described in the last section, underlies the fishnet model, the "good" three-scalar model underlies the strongly-β-twisted model as we will see later on. We hence claim that it is worth understanding the good three-scalar sector as well as possible. Thus, we turn our attention to this sector in the following. There are six different three-scalar sectors, which fall under the definition "good", namely those which contain all three flavours and at least one field and one conjugate field. The example that we have used so far and that we will keep using contains operators including ϕ_1, ϕ_2 and $\phi_3^\dagger$. Notice further that if we exchange the number of ϕ_1 and ϕ_2 and take ϕ_2 as the vacuum, we get the mirrored spin-chain, i.e., the spin-chain where all right-movers have become left-movers and transmission during scattering happens in the opposite direction. This spin-chain necessarily has the same Hamiltonian (in a different basis). We can, therefore, restrict ourselves in principle to the cases, where we have more ϕ_1 then ϕ_2 fields in the local operators.

We have derived the corresponding Bethe equations in section 3.3.1. They are given in (3.68) to (3.70) and can be extended with the zero-momentum condition. We use the dualised version of the Bethe equations, which are

$$(-1)^{K-1} = \prod_{j=1}^{M} (\alpha_{3,k} - \alpha_{4,j}) \tag{4.7}$$

$$\alpha_{4,k}^L = (-1)^{K_4-1} \prod_{j=1}^{K_4-K_3} \frac{1}{\alpha_{4,k} - \alpha_{3,j}} \tag{4.8}$$

$$E = \sum_{k=1}^{M} \frac{-2}{\alpha_{4,k}}, \tag{4.9}$$

$$\prod_{j=1}^{M} \alpha_{4,j} = 1\,, \tag{4.10}$$

sum of three roots of unity.

where as stated before we have $(L-M)$ ϕ_1, $(M-K)$ ϕ_2 and K $\phi_3^\dagger$ and E is the eigenvalue of the Hamiltonian.

The questions we want to answer in this section are: 'What are the solutions to (4.7) to (4.10)?' and 'Do these equations provide a complete eigenbasis of the Hamiltonian?'. The one magnon solution is trivial. It is the same as in the two-scalar sector since if we have only one excitation, we necessarily restrict ourselves to having only two scalars. Also with the zero-momentum condition there are only two one-magnon states, one for a ϕ_2 and one for a $\phi_3^\dagger$ excitation, each of which only picks up a factor of -2 under the action of the Hamiltonian.

The first interesting new case contains two magnons, one ϕ_2 and one $\phi_3^\dagger$. Indeed, the Bethe equations for $M=2$ and $K=1$ corresponding to these excitations can be solved exactly for arbitrary length L. One of the Bethe roots is from one of the following two sets

$$\alpha_{4,1} \in \{\exp((2k+1)i\pi/(L+1)) | 0 \leq k \leq (L-1)/2\} \text{ or} \tag{4.11}$$
$$\alpha_{4,1} \in \{\exp(2ki\pi/(L-1)) | 1 \leq k \leq (L-2)/2\}. \tag{4.12}$$

Due to the cyclicity constraint the other Bethe root is simply its complex conjugate. These roots lead to the energies

$$E = -4\cos((2k+1)\pi/(L+1)) \qquad \text{for } 0 \leq k \leq (L-1)/2 \text{ and} \tag{4.13}$$
$$E = -4\cos(2k\pi/(L-1)) \qquad \text{for } 1 \leq k \leq (L-2)/2, \tag{4.14}$$

respectively. For the generic case, the two sets of possible Bethe roots are disjoint, however, when $L-1$ is divisible by four the Bethe root $\alpha_{4,1} = i$ appears in both sets. In this case, we have one solution to the Bethe equations less than expected, i.e., we do not find a complete set of eigenstates of the Hamiltonian anymore. In fact, the Hamiltonian is not diagonalisable anymore and we can only bring it into Jordan normal form. This is the first time we see explicitly that the Hamiltonians for these models are non-diagonalisable even for non-eclectic field content. We call this type of Jordan block accidental, since it depends on the cancellation of non-zero coefficients, in contrast to those from the eclectic sector.

In this simple case, we can explicitly work out the Jordan block. To investigate this further let us label the states by how far the ϕ_2 is to the right of the $\phi_3^\dagger$ as $|1\rangle = |\phi_3^\dagger\phi_2\phi_1\cdots\rangle$, $|2\rangle = |\phi_3^\dagger\phi_1\phi_2\phi_1\cdots\rangle$ and so on. Then for $L=5$ the eigenstate, the top, of the Jordan block is given by

$$|\Psi_{\text{eigen}}\rangle = |4\rangle - |2\rangle\,. \tag{4.15}$$

This eigenstate corresponds to the pair of Bethe roots $\alpha_{4,1} = i$ and $\alpha_{4,1} = -i$ and accordingly has the eigenvalue $E=0$. The second state of the Jordan block of size two is

$$|\Psi_{\text{JB}}\rangle = \frac{1}{-2}\left(|3\rangle - 2\,|1\rangle\right) + c\,|\Psi_{\text{eigen}}\rangle\,, \tag{4.16}$$

where c is an arbitrary constant. We can add any multiple of $|\Psi_{\text{eigen}}\rangle$ to the second state of the Jordan block since the eigenstate is annihilated by the Hamiltonian. We can repeat the same analysis for the case $L=9$, where the eigenstates will be a wave as above, while the coefficients of the second state continue to grow linearly, with alternating signs. This means for $L=9$ the generalised eigenvector is

$$|\Psi_{\text{JB}}\rangle = \frac{1}{-2}\left(|7\rangle - 2\,|5\rangle + 3\,|3\rangle - 4\,|1\rangle\right)\,, \tag{4.17}$$

where as before we can add any multiple of the corresponding eigenstate. These non-eigenstates are not of the form given by our ansatz and hence can also not be found by a Bethe ansatz in the usual way.

As a next step, we want to add additional magnons. The Bethe equations are simpler if we restrict ourselves to just one $\phi_3^\dagger$ or equivalently one auxiliary root. Thus let us add an additional ϕ_2. The Bethe equations are too complicated to be solved explicitly for this case, in general. However, specific solutions can be found and will be presented below. Furthermore, we can explicitly diagonalise the spin-chain for small length L. We delay the presentation of the approximate solutions to section 4.4, where they appear as a subsector of the maximal non-eclectic sector. Interestingly, we find Jordan blocks of size two for length $L = 5 + 3k$, with k being a non-negative integer.

In order to analyse the Jordan blocks, we first have to find a notation to label the states. We count how many spin-chain sites the two ϕ_2 excitations are to the right of the $\phi_3^\dagger$. For example, the state $|1,3\rangle$ has two ϕ_2 excitations, one immediately following the $\phi_3^\dagger$ and the other one three spin-chain sites to the right of $\phi_3^\dagger$. We now discuss the action of the Hamiltonian on a state $|m,n\rangle$. Either H acts on the $\phi_3^\dagger$, which decreases m and n by 1 or it acts on one of the ϕ_2 increasing either m or n by 1. If, $n = L-1$ there is also the possibility that H moves the ϕ_2 past the $\phi_3^\dagger$ creating the state $|1, m+1\rangle$. In particular, if $L-2$ is divisible by 3, which is the case when Jordan blocks appear, the Hamiltonian changes the sum of $m+n$ to $m+n+1$ mod 3. For these specific spin-chain lengths, the Hamiltonian cyclically moves through three different subspaces of states, distinguished by the remainder of $m+n$ divided by 3. All of these subspaces have the same dimension.

For length $L = 5 + 3k$ there are eigenstates with eigenvalue zero in two of these subspaces, let us call them V_1 and V_2, although using Mathematica we are only able to find one of the two eigenstates from the Bethe equations. Furthermore, it appears that the third subspace, V_3, does not contain an eigenstate of eigenvalue zero. This then implies that there has to be a Jordan block of at least size two because eigenstates of the Hamiltonian can not contain any part of V_3, which are not in the image of V_2 (or V_1) under H. Consequently, the dimension of the vector space spanned by all the eigenvectors is $dim(V_1) + dim(V_2) + dim(V_3) - 1$, which shows that the Hamiltonian is defective. Unfortunately, while we can explicitly construct two eigenstates with eigenvalue zero in V_1 and V_2, there is so far no proof that V_3 can not contain one eigenvector with eigenvalue zero. On the other hand, we can determine the Jordan block explicitly as done in the following paragraphs.

For $L = 5 + 3k$, there is an eigenstate, with eigenvalue zero, which is given by the following superposition. In the superposition all states $|1+3r, 3+3s\rangle$ appear, with a coefficient of 1 and the states $|3+3r, 4+3s\rangle$ appear with coefficient -1. Here, r and s are all non-negative integers, such that the second entry in the ket is larger than the first but smaller than L. All these states fullfill $m+n = 1$ mod 3 and $n-m \neq 3$ mod 3. This is the top of the Jordan block appearing.

The second state of the Jordan block is given by a superposition of words of the following form

$$\Psi_{\text{Jordan}} = \sum_{n=2}^{L-1} \sum_{m=1}^{n-1} c_{m,n} \, |m,n\rangle \ . \tag{4.18}$$

Here, the coefficients $c_{m,n}$ can indeed be completely determined. For $n+m \neq 0$ mod 3 the coefficients are zero. For $m = 1$ mod 3 we have

$$c_{3m'-2,3k+5-3n'} = \frac{-n'}{k+1}(3k - n' - 2m' + 6) \, . \tag{4.19}$$

For $m = 2 \bmod 3$ and $m = 0 \bmod 3$ we have

$$c_{3m'-1,3k+7-3n'} = \frac{(k-m'+2)(2n'+m'-2)}{k+1}, \tag{4.20}$$

$$c_{3m',3k+6-3n'} = \frac{(m'-n')(m'+n'-k-2)}{k+1} \tag{4.21}$$

respectively.[2] With some persistence, one can plug these coefficients into the formula for the Jordan block (4.18), act with H and check that one indeed obtains the eigenstate with eigenvalue zero described in the previous paragraph. We know that there has to be a Jordan block now since otherwise (4.18) could be expanded in eigenstates of the Hamiltonian. Acting with H^2 would then multiply each coefficient in the expansion by the eigenvalue squared λ_i^2. But since the basis of eigenvectors is linearly independent and $H^2\Psi_{\text{Jordan}} = 0$ all eigenvalues appearing in the expansion would have to be zero, which is not the case since $H\Psi_{\text{Jordan}} \neq 0$. Thus, there is no eigenbasis for the Hamiltonian, i.e., the matrix is defective.

Similar to the two-magnon case described earlier, we have the largest absolute value of a coefficient for the word $|1,2\rangle$, with mostly decreasing coefficients for increasing n. This increase/decrease in coefficients as we move once around the spin-chain seems typical for the generalised eigenstates of the Jordan blocks. It is related to the fact that the $\phi_3^\dagger$ is only letting the ϕ_2 move from position $L-1$ to position 1 and not the other way around, hence there is no backreaction of the state $|1,2\rangle$ to $|1,n\rangle$, i.e., acting with H will only take you from the latter to the former and not from the former to the latter. This kind of semi-wall appears to be critical for the formation of the Jordan blocks. A natural assumption is that this could explain why the generalised eigenvalues are zero as well.

Going to four magnons, with three ϕ_2 excitations and one $\phi_3^\dagger$ excitation we might expect from the Bethe equations that we again find a Jordan block at all odd L. Strictly speaking, we can only claim we find a solution to the Bethe equations with energy zero. This solution has the auxiliary root being zero, while the momentum-carrying roots are two complex conjugate pairs, which are related to each other through multiplication by -1. Indeed, one, two and three Jordan blocks appear for length five, nine and 13, respectively. Furthermore, there are two Jordan blocks appearing for length seven and three for length eleven. All of these Jordan blocks are only of size two, an interesting feature that still eludes explanation. In addition, all of the Jordan blocks have generalised eigenvalues zero. However, for higher length $L \geq 9$ Jordan blocks with different generalised eigenvalues may exist, they might just be computationally too expensive to find. The matrices reach a size for which it is difficult to bring them in Jordan normal form but to find the nullspace (of H, H^2 and H^3) is still readily available.

Generalising the above to magnon numbers $K = 1$ and $M = 4n$, for any integer n is straight-forward. The Bethe equations have the zero-energy solution described above for all odd length $L \geq M+1$. For the remaining even magnon numbers, the situation only appears if i and $-i$ are also valid Bethe roots. This implies the zero-energy solution appears only in steps of four at length $L = 5+4k$, as long as $L \geq M+1$. For odd magnon numbers, the situation is significantly more involved because the cancellation of the Bethe roots does not happen pairwise, but between all Bethe roots taken together. The zero-energy solution to the Bethe equations exists, when the Bethe roots are $\exp(2n\pi i/M)$, i.e., the Bethe roots are equally distributed around the unit circle. The Bethe roots above are valid, if $L+1$ is divisible by M and $L \geq M+1$. At this point, it is natural to conjecture that all

[2]Special thanks to Julian Miczajka for his collaboration regarding the derivation of these general formulas.

these zero-energy solutions are indeed a part of a Jordan block of size two with generalised eigenvalue zero. We checked systematically that indeed Jordan blocks appear up to length nine. Some specific examples were found for any length, see the discussion around the three magnon case above.

The natural next question to ask is what happens if we include more $\phi_3^\dagger$ excitations. Unfortunately, in this case, we know very little since we are introducing more auxiliary roots. As a consequence, the auxiliary roots can not be zero anymore and appear in the Bethe equations. It becomes significantly more difficult to solve the Bethe equations. Furthermore, there tend to be more distinct states at a given length and hence the Hamiltonian for a given length is a larger matrix, which poses a computational obstacle for diagonalisation. We found results up to length seven. No Jordan blocks appeared, and the energies are complex numbers, with absolute values somewhere in the range between 0.1 and ten.

4.3 Two Scalars with Derivatives

In this section, we turn our attention back to the fishnet theory. In particular, we analyse the Bethe equations for the sector containing two scalars and derivatives. Let us remind ourselves of the Bethe equations, they are

$$1 = \prod_{j=1}^{K_R} \alpha_{4,j} \prod_{k=1}^{K_\partial} \frac{\tilde{u}_{4,k} + i/2}{\tilde{u}_{4,k} - i/2} \tag{4.22}$$

$$\alpha_{4,k}^L = (-1)^{K_R - 1} \prod_{j=1}^{K_\partial} \frac{(\tilde{u}_{4,j} + 3i/2)(\tilde{u}_{4,j} - i/2)}{(\tilde{u}_{4,j} + i/2)(\tilde{u}_{4,j} + i/2)}\,, \tag{4.23}$$

$$1 = \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_\partial} \frac{u_{2,k} - \tilde{u}_{4,j}}{u_{2,k} - \tilde{u}_{4,j} - i}\,, \tag{4.24}$$

$$1 = \prod_{j \neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_\partial} \frac{u_{6,k} - \tilde{u}_{4,j}}{u_{6,k} - \tilde{u}_{4,j} - i}\,, \tag{4.25}$$

$$\begin{aligned} \left(\frac{\tilde{u}_{4,k} + i/2}{\tilde{u}_{4,k} - i/2}\right)^{L - K_R} &= \left(\frac{\tilde{u}_{4,k} + i/2}{\tilde{u}_{4,k} + 3i/2}\right)^{K_R} \prod_{j \neq k}^{K_\partial} \frac{\tilde{u}_{4,k} - \tilde{u}_{4,j} - i}{\tilde{u}_{4,k} - \tilde{u}_{4,j} + i} \\ &\quad \times \prod_{j=1}^{K_2} \frac{\tilde{u}_{4,k} - u_{2,j} + i}{\tilde{u}_{4,k} - u_{2,j}} \prod_{j=1}^{K_6} \frac{\tilde{u}_{4,k} - u_{6,j} + i}{\tilde{u}_{4,k} - u_{6,j}}\,, \end{aligned} \tag{4.26}$$

where we take K_R to be the number of ϕ_2 excitations and K_∂ to be the number of derivatives. K_2 and K_6 change the type of derivatives.

The Bethe equations in this sector show a fascinating structure. If we ignore the zero-momentum condition, for the time being, they seem to decouple in the sense that the Bethe roots for the ϕ_2 do not enter the equations for the remaining roots. Thus, in principle, we can solve equations (4.24) to (4.26) for the $\tilde{u}_{4,k}$ roots and plug the result into equation (4.23). After doing so, we are left effectively with the two-scalar equations, which contain some number that is determined by the roots corresponding to the derivatives. This supports our earlier statement that the two-scalar problem underlies the whole fishnet-theory. The effect of the derivatives is independent of the distribution of the ϕ_2 excitations as long as the zero-momentum condition is ignored. However, interestingly, once the zero-momentum condition is imposed, the effect of the derivatives changes dramatically as we will see in the explicit solutions we are going to discuss later.

It is also interesting to note that the equations describing the derivatives (4.24) to (4.26) look very similar to the standard $\mathfrak{su}(2,2)$ equations. In fact, we can make the similarity even clearer by shifting

$$u_{6,k} \to u_{6,k} + \frac{i}{2} \quad \text{and} \quad u_{2,k} \to u_{2,k} + \frac{i}{2}\,. \tag{4.27}$$

This transformation will yield the typical scattering terms

$$\frac{\tilde{u}_{4,k} - u_{6,j} + i/2}{\tilde{u}_{4,k} - u_{6,j} - i/2}$$

and similarly for u_2. The only difference then becomes in the non-scattering term

$$\left(\frac{\tilde{u}_{4,k} + i/2}{\tilde{u}_{4,k} - i/2}\right)^{L-K_R} \left(\frac{\tilde{u}_{4,k} + 3i/2}{\tilde{u}_{4,k} + i/2}\right)^{K_R}\,,$$

for which exactly K_R of the $\tilde{u}_4$ enter with shifts. Usually, this term is connected to the representation of the spin-chain. Thus, it is natural to conjecture that the $\mathfrak{su}(2,2)$ spin-chain consists of spin-chain sites with vector spaces of two different representations depending on whether the underlying field is ϕ_1 or ϕ_2. To show the concrete relations is left for future work, however, it strengthens the confidence that the Bethe equations are correct, as otherwise, this would be quite a coincidence.

As argued above the Bethe equations with derivatives are based on those of the two-scalar sector. As a consequence, the Bethe equations are simpler than that of $\mathcal{N} = 4$ SYM. They are, however, still more complicated than those of the two-scalar sector. In particular, one problem that has increased significantly is that of counting the solutions to the Bethe equations. While in principle, one might expect that for each solution of (4.24) to (4.26) one will find the appropriate number of solutions for (4.23), the question arises what one does at special values of $\tilde{u}_{4,k}$. Does one allow solutions that have $\tilde{u}_{4,k} \in \{3i/2, i/2, -i/2\}$, where certain factors of the Bethe equations are zero or diverge? If the answer to the above question is affirmative, because one explicitly finds corresponding eigenstates, then one can only have one solution to (4.23). As a result, the Bethe equations do not have enough solutions to yield the full spectrum anymore, which might lead to the assumptions that Jordan blocks have formed.

To explicate all of the above let us look at a simple example. Let us consider the spin-chain of length L with exactly one ϕ_2 and one derivative. The Bethe equations for this chain can be reduced to the polynomial equation

$$2\left(\frac{E}{-2}\right)^{L-1} - \left(\frac{E}{-2}\right)^{L-2} - 1 = 0\,, \tag{4.28}$$

where E is the energy. Since this is a polynomial equation of degree $L-1$, but the Hilbert space of the spin-chain is L-dimensional we are clearly missing one solution. The missing eigenstate is

$$|(\partial\phi_2)\phi_1^{L-1}\rangle - |\phi_2(\partial\phi_1)\phi_1^{L-2}\rangle\,. \tag{4.29}$$

This state gets annihilated by H, i.e., $E = 0$, because

$$H\,|(\partial\phi_2)\phi_1^{L-1}\rangle = H\,|\phi_2(\partial\phi_1)\phi_1^{L-2}\rangle\,. \tag{4.30}$$

In fact, the above combination of derivatives and ϕ_2 presents an obstacle for potential other excitations and we hence call it a wall. We describe these walls in appendix C. Since

these walls describe insurmountable obstacles for other ϕ_2 excitations, we conclude that adding additional ϕ_2 to the operator will lead to a Jordan block, which will be roughly of size L.

Above we claimed that one eigenstate of the spin-chain is missing, however, we can find this eigenstate from the Bethe equations if we allow some factors to diverge. In particular, if we take $\tilde{u}_4 = -i/2$, then equation (4.26) in its current form is satisfied. Plugging this Bethe root into (4.23) will cause the right-hand side to diverge, forcing α to diverge and leading to a zero-energy solution. Since α appears as a coefficient multiplying states in a vector space, it should not be infinite, and the interpretation of this solution is unclear. However, notice that under the assumption that the S-matrix is zero the only two states that can enter the eigenstate are the ones related by S-matrix, which is indeed the case. This brute force way of finding a solution to the Bethe equation works, although the ansatz leading to the equations breaks down.

The remaining solutions converge to the solutions of the spin-chain of length $L-1$ without the derivative and without the cyclic invariance imposed in the $L \to \infty$ limit. It is easy to see from the polynomial equation for the energy that $|E/2|$ has to be close to one. Closer inspection shows that the solutions behave like

$$-\frac{E}{2} = \frac{e^{2\pi i n/(L-1)}}{(2 - e^{-2\pi i n/(L-1)})^{\frac{1}{L-1}}} + \mathcal{O}(L^{-2}) \tag{4.31}$$

in the large L limit. This agrees with the solutions for the spin-chain of length $L-1$ without derivatives up to terms of order $1/L^2$. To some degree, this had to be expected because the Hamiltonian is of nearest neighbour type and the derivatives do not travel around the chain by themselves. Therefore, the derivative is stationary, when the ϕ_2 is far away, which are nearly all states in the large L-limit. Consequently, it does not appear to be a real excitation, but rather a reference point to which the distance of the ϕ_2 can be measured and hence cyclic invariance is also lost. When the derivative and the ϕ_2 are close, the ϕ_2 excitation can catalyse the movement of the derivative. This implies that the reference point, the derivative, is not completely stationary. The boundary conditions are slightly altered and the spin-chain length is effectively shortened.

The energies of the states with lowest lying classical dimensions, explicitly $\Delta_0 \leq 5$, are shown in table 4.1, while $\Delta_0 = 6$ is given in appendix D. For states with higher Δ_0, there are too many eigenstates to be shown in a reasonable table. All eigenvalues up to $\Delta_0 \leq 7$ are plotted in the complex plane in figure 4.2. We see from table 4.1 that we find the energies of several descendants, which have the same anomalous dimension as the primary, but their classical dimensions are higher by an integer. For example the primaries

$$O(x)_\pm = \pm\sqrt{2}\,\mathrm{tr}(\phi_1\phi_1\phi_2\phi_2) + \mathrm{tr}(\phi_1\phi_2\phi_1\phi_2) \tag{4.32}$$

with energies $E = \mp\sqrt{2}$ appears at $\Delta_0 = 4$. Adding one and two derivatives descendants appear at $\Delta_0 = 5$ and $\Delta_0 = 6$ respectively, with the same energy. The anomalous dimensions of these primaries were already found in [54].

However, adding derivatives does not just lead to descendants, but also yields new primaries. For example, for operators with two ϕ_1, two ϕ_2 and one derivative, we find new primaries, with zero anomalous dimension and Jordan blocks of size three. Adding another derivate, even more primaries appear with energies $E = \pm 2i$ and $E = \pm 1.15$.

Δ_0	ϕ_1	ϕ_2	∂	E	multiplicity
3	2	1	0	-2	1
4	2	1	1	-2	4
				0	4
				1	4
	3	1	0	-2	1
	2	2	0	$-2\sqrt{2}$	1
				$2\sqrt{2}$	1
5	2	1	2	-2	10
				-2/3	9
				0	40
				1	16
	3	1	1	-2	4
				0	4
				$1/2 + i\sqrt{7}/2$	4
				$1/2 - i\sqrt{7}/2$	4
	2	2	1	0	4
				$-2\sqrt{2}$	4
				$2\sqrt{2}$	4
				JB(0,3)	4
	4	1	0	-2	1
	3	2	0	$-1 - \sqrt{5}$	1
				$-1 + \sqrt{5}$	1

Table 4.1: The energies of the states with lowest lying classical dimension and $L \geq 3$ in the fishnet theory are listed. Eclectic states and states with only one type of scalar are not shown. Furthermore, only states with at least as many ϕ_1 as ϕ_2 are shown. JB(0,3) stands for a Jordan block with generalised eigenvalue zero and of size three.

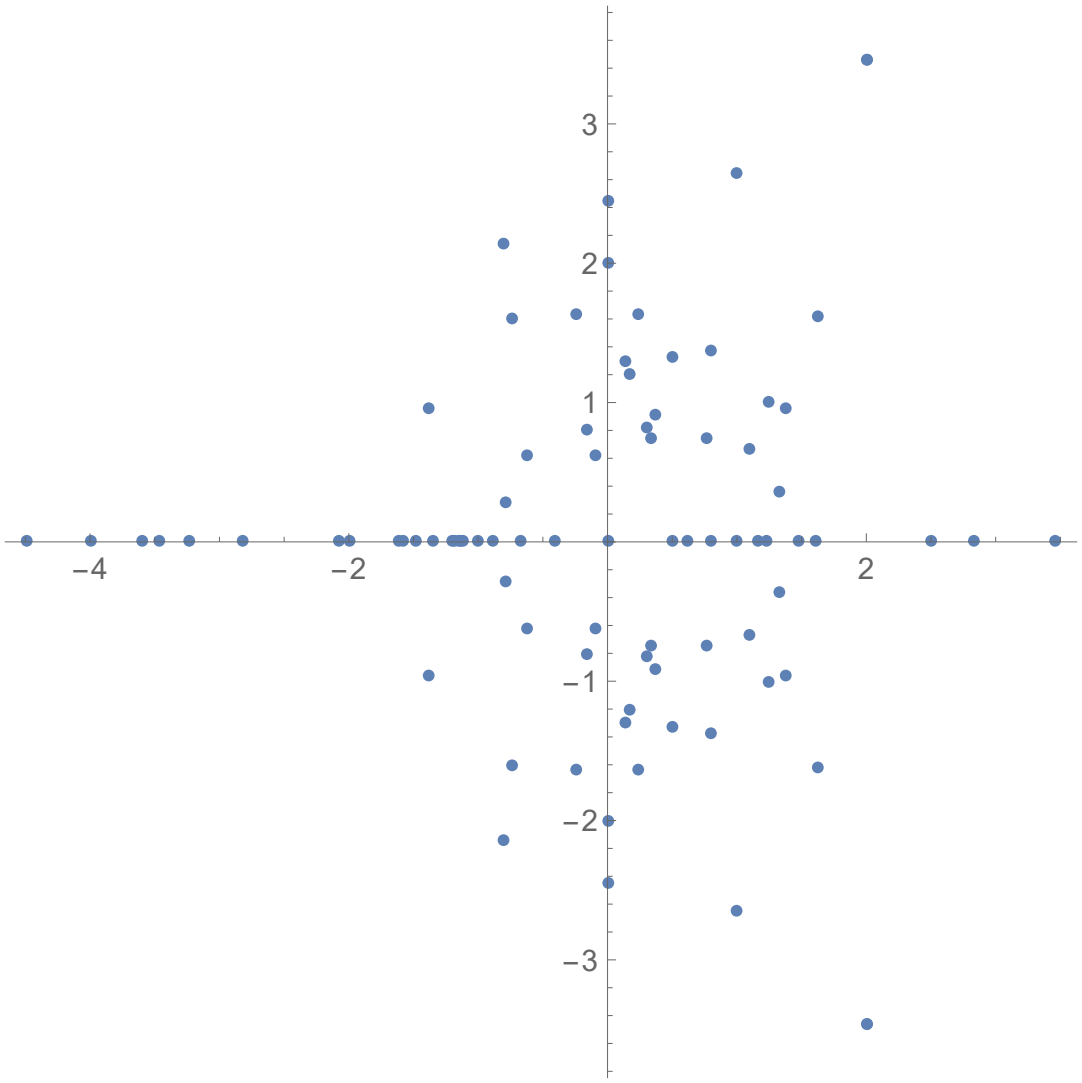

Figure 4.2: Energies of states with $\Delta_0 \leq 7$ in the fishnet model.

4.4 The non-Eclectic Sector of the β-Deformation

In this section, we take a look at the form of the Bethe equations for the strongly-β-twisted theory. Similarly to what we observed in the fishnet theory, we see a curious decoupling of the Bethe equations into two sets of equations. The corresponding Bethe roots can be split into two groups. We call category one Bethe roots those that change the nature of the excitations between left- and right-movers and non-dispersing excitations, on any level of nesting. Category one Bethe roots do not affect the category two roots, which change the excitations only within one type, for example, left-movers into left-movers. Again, we may attempt to solve for the independent set of Bethe roots first and then plug the results into the remaining equations. The decoupling of the equations is the same as in the case of the fishnet model. However, we can now include right-movers at both levels of nesting, such that there are still two types of category one roots and the problem reduces to that of the three-scalar sector with some additional factors. Again, these factors are constant once a valid solution has been found for the category two roots.

Let us illustrate the above slightly technical analysis on the three sectors, for which we have determined the Bethe equations in chapter 3. A good starting point is to add the two fermions $\bar{\Psi}_{3\dot{1}}$ and $\bar{\Psi}_{3\dot{2}}$ to the three-scalar model. The corresponding Bethe equations were worked out in section 3.5.2 and are given by (3.100) to (3.105). From equations (3.101) and (3.102), we can determine that

$$C^{K_3} := \left(\prod_{j=1}^{K_2} \frac{u_{2,j}+i/2}{u_{2,j}-i/2}\right)^{K_3} = 1\,. \tag{4.33}$$

Plugging this result into (3.103) leads to the three-scalar equation, with an additional factor C, i.e., (3.103) becomes

$$1 = (-1)^{K_3-1} \prod_{j=1}^{K_4} (\alpha_{4,k} - \alpha_{3,j})\, C\,, \tag{4.34}$$

while (3.100), (3.104) and (3.105) are already the same as the three-scalar equations.

The decoupling of the Bethe equations is not that surprising from some points of view. Naively, the twisting of $\mathcal{N}=4$ SYM only affects the flavour parts; fields with identical flavour get twisted in the same way. Therefore it is not surprising that Bethe roots, corresponding to no change in the flavour of the excitations, are hardly affected by the twisting. More precisely, the Hamiltonian of the strongly-twisted models is basically unable to differentiate between a $\phi_3^\dagger$ and a $\bar{\psi}_3$ excitation[3]. Its action on the two excitations are identical. However, the states differing by a $\phi_3^\dagger$ and a $\bar{\psi}_3$ are still two different states, and accordingly, the eigenvectors of the three-scalar sector are different once the fermionic excitations are added. Let us consider, for example, the subspaces containing a ϕ_1 and two $\phi_3^\dagger$, or a ϕ_1, a $\bar{\psi}_3$ and a $\phi_3^\dagger$. In the first case, we have only one single state $|\phi_1\phi_3^\dagger\phi_3^\dagger\rangle$ and all others are just cyclic permutations. It is an eigenvector of the Hamiltonian with eigenvalue -2. Since in the second case the excitations with the $\bar{3}$ flavour are distinguishable, we get two different states $|1\rangle = |\phi_1\phi_3^\dagger\bar{\psi}_3\rangle$ and $|2\rangle = |\phi_1\bar{\psi}_3\phi_3^\dagger\rangle$. Consequently, there are also two different eigenvectors $|1\rangle + |2\rangle$ and $|1\rangle - |2\rangle$. The fact that the distinguishability of excitations is left untouched by the twisting explains why (3.100) and (3.101) resemble those of untwisted $\mathcal{N}=4$.

[3]We do not specify which component of the fermion is taken either $\bar{\psi}_{3\dot{1}}$ or $\bar{\psi}_{3\dot{2}}$ would work.

The second sector we look at in the strongly-β-twisted theory is the one containing the fields

$$A \in \{\partial^k \phi_1, \partial^k \phi_3^\dagger, \partial^k \psi_1, \partial^k \bar{\psi}_3\} \tag{4.35}$$

where ∂^k stands schematically for any product of the four derivatives and we suppressed the spinor indices on the fermions, but allow them to take both values. The Bethe equations for this sector are given in (3.107) to (3.111). A critical observations is that this sector is significantly different from the sector, where we include instead of $\phi_3^\dagger$ and $\bar{\psi}_3$, ϕ_2 and ψ_2, due to the process

$$(\partial \phi_3^\dagger)\phi_1 \to \psi_1 \bar{\psi}_3 \,, \tag{4.36}$$

which has no equivalent in a sector without antifermions. Following the logic above, the Bethe equations in this sector should decouple into two sets. The first one can be solved without knowledge of the remaining Bethe equations. Since we only have two different flavours in this sector, 1 and $\bar{3}$, the remaining Bethe equations should then be transformed into the two-scalar equations times a factor determined by the decoupled equations. Indeed we see that we can eliminate all Bethe roots apart from the α_4 and one constant and obtain one single equation

$$\alpha_k^L = (-1)^{K_R} C \,, \tag{4.37}$$

where C is independent of α and is given by some rational function of the $\tilde{u}_4$ and $\tilde{u}_3$.

Finally, we also consider the full non-eclectic sector of the β-twisted theory. It contains a maximum number of different fields compatible with being non-eclectic. Explicitly, we have the three-scalar model with the corresponding fermions and all four derivatives:

$$A \in \{\partial^k \phi_1, \partial^k \phi_2, \partial^k \phi_3^\dagger, \partial^k \psi_1, \partial^k \psi_2, \partial^k \bar{\psi}_3\}. \tag{4.38}$$

The proposed Bethe equations are given in (3.117) to (3.122). (3.119) to (3.122) are independent of the first two equations. They are the Bethe equations of a $\mathfrak{su}(2,2)$ spin-chain with spin-chain sites of two different representations. Given the solution to this $\mathfrak{su}(2,2)$ spin-chain, one can plug the roots into the remaining Bethe equations to reduce them to a three-scalar model with two additional factors. This time, in contrast to the three-scalar plus two fermions model, both equations get multiplied by an additional factor, which had to be expected since we allow for excitations with the same flavour as the vacuum, i.e., our operators are allowed to contain ψ_1 fields.

The energies of the states with lowest classical dimension Δ_0 under the condition that $L \geq 3$ are shown in table 4.2 and figure 4.3. The β-twisted theory contains the largest non-eclectic sector of the fishnet theory $\{\partial^k \phi_1, \partial^k \phi_2\}$ as a subsector and hence all the energies of the fishnet theory will reappear in the β-twisted theory. We see many of the energies repeated for the states obtained when replacing a scalar by a fermion of the same flavour. This is due to the fact described above that the Hamiltonian only sees the flavour of a field and not its bosonic or fermionic nature. Furthermore, when exchanging a scalar with a fermion, we can often create an antisymmetric state with zero energy, which explains why this specific eigenvalue appears ubiquitously. Interestingly, introducing fermions also creates Jordan blocks. This phenomenon occurs for the first time for $\Delta_0 = 4.5$. It is similar to the derivative type wall described in appendix C, in so far as we can built the combination

$$|\psi_2 \phi_1 \phi_3^\dagger\rangle - |\phi_2 \psi_1 \phi_3^\dagger\rangle \,, \tag{4.39}$$

which is annihilated by the Hamiltonian. Here, the bosonic or fermionic nature plays the role of the derivative in distinguishing the two states. Since the Hamiltonian acts on both states in the same way, the minus sign makes sure that the linear combination is

Δ_0	$a_1^\dagger$	$a_2^\dagger$	$b_{\dot{1}}^\dagger$	$b_{\dot{2}}^\dagger$	$c_1^\dagger$	$c_2^\dagger$	$c_3^\dagger$	$c_4^\dagger$	E
3	0	0	0	0	2	1	0	3	-2
					2	2	0	2	$\pm 2\sqrt{2}$
3.5	1	0	0	0	2	1	0	2	$\{-2, 0, 1\}$
					2	2	0	1	$\{\pm 2\sqrt{2}, \mathrm{JB}(0,2)\}$
					3	1	0	1	± 2
					3	2	0	0	-2
	0	0	1	0	3	1	0	3	-2
					2	2	0	3	$\pm 2\sqrt{2}$
					3	2	0	2	± 2
4	0	0	0	0	3	1	0	4	-2
					2	2	0	4	$\pm 2\sqrt{2}$
					3	2	0	3	$\{-1 \pm \sqrt{5}, 2\}$
					3	3	0	2	$\{-3.17, 1.59 \pm 2.75i\}$
	2	0	0	0	2	1	0	1	$\{0, 0, 1\}$
					3	1	0	0	2
					2	2	0	0	JB(0,2)
	1	1	0	0	2	1	0	1	$\{-2, 0, 0, 0, 1, 1\}$
					3	1	0	1	± 2
					2	2	0	0	$\{\pm 2\sqrt{2}, \mathrm{JB}(0,2)\}$
	0	0	2	0	3	2	0	3	2
	0	0	1	1	3	2	0	3	± 2
	1	0	1	0	2	1	0	3	$\{-2, 0, 1\}$
					3	1	0	2	$\{-2, -2, 2, 1/2 \pm \sqrt{7}i/2\}$
					2	2	0	2	$\{\pm 2\sqrt{2}, \pm 2\sqrt{2}, \mathrm{JB}(0,2), \mathrm{JB}(0,2), \mathrm{JB}(0,2)\}$
					3	2	0	1	$\{-2, -2, 2, 1/2 \pm \sqrt{7}i/2\}$

Table 4.2: Energies of states with classical dimension $\Delta_0 \leq 4$ and $L \geq 3$ in the strongly-β-twisted theory are listed. States related by interchanging the number of ϕ_1 and ϕ_2 or by picking an equivalent set of spinor indices are not shown.

annihilated. Furthermore, the ϕ_2 is the ultimate right mover, in the sense that nothing can be transmitted through it in the right direction and hence the above combination of ϕ_1, ψ_1, ϕ_2 and ψ_2 forms a wall for all other excitations. As a consequence, there are Jordan blocks of arbitrary large size in the β-twisted theory involving the derivative wall states described in appendix C. So far we have only found Jordan blocks with generalised eigenvalue zero, however, there has not been a convincing argument, why non-zero generalised eigenvalues should be forbidden. In the next section, we will discuss some results concerning the Jordan blocks.

4.5 Jordan Blocks and Logarithmic CFTs.

In this section, we will look at arguably the most important impact of non-unitarity in the theories discussed: the emergence of Jordan blocks. In a unitary CFT, the dilatation operator is hermitian, resulting in two critical features. Firstly, the eigenvalues of hermitian operators are real. In the context of the dilatation operator, the implication is that the anomalous dimensions are real. Secondly, hermitian operators are diagonalisable, while for non-hermitian operators, the best we can do in general is to put them in Jordan

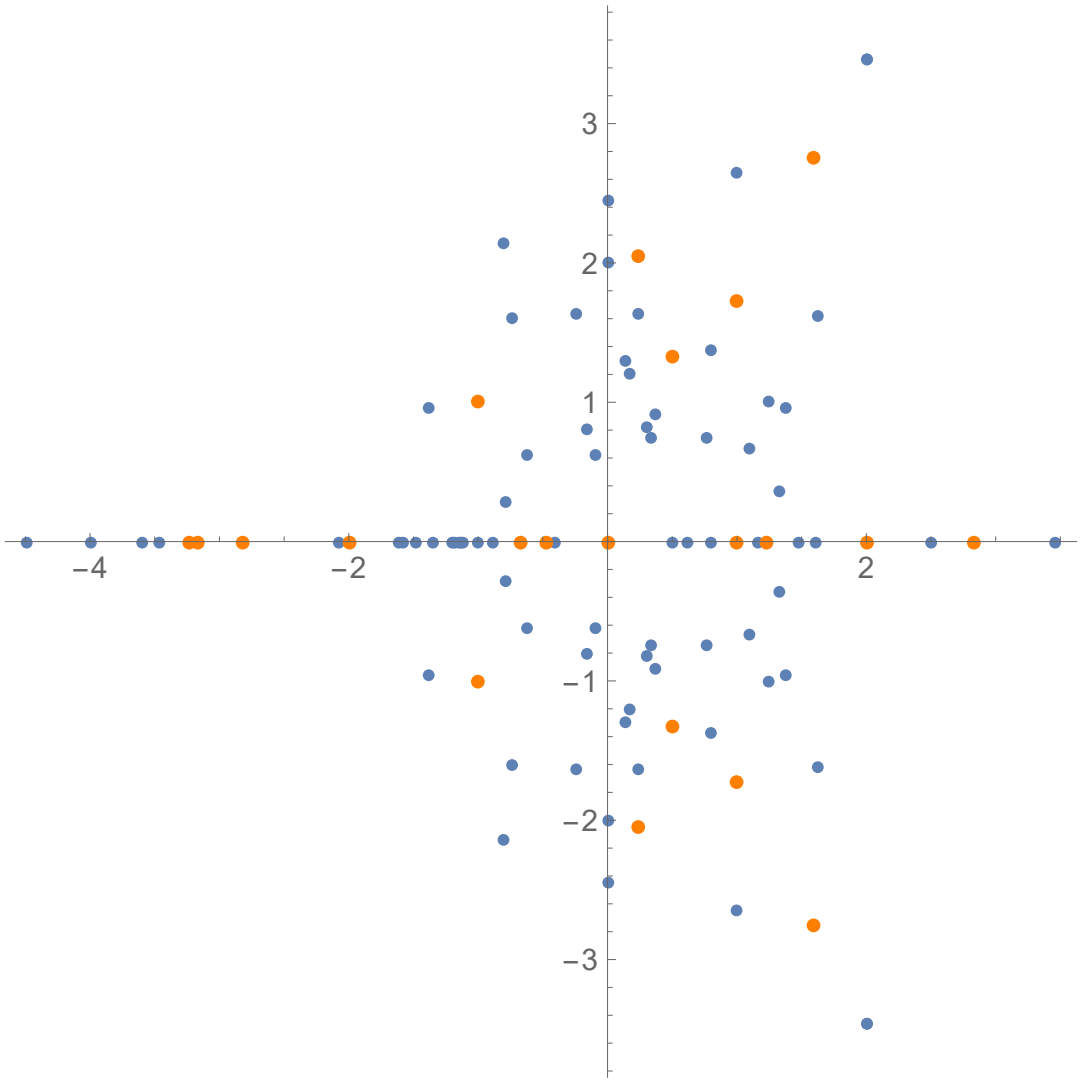

Figure 4.3: Energies of states with $\Delta_0 \leq 4.5$ in the strongly-β-twisted model are shown in orange in the complex plane. In addition in blue we show the energies of the maximal non-eclectic part of the fishnet model for $\Delta_0 \leq 7$. Since this sector is also a subsector of the strongly-β-twisted theory they are also energies of states in the β-twisted theory.

normal form. An important consequence is the change in the form of two-point correlation functions in the theory as discussed briefly in section 2.1.4 and 2.3.2.

So far, we have calculated the dilatation operator, assuming the standard form of the two-point functions known from unitary CFTs. Since the logarithmic two-point functions are not of the standard form, this assumption does not hold. However, our derivation of the dilatation operator is still valid, as we will show in the following. In 2.1.4, we have made another critical assumption, which turns out to be invalid as well, namely that the operators and their conjugates are subject to the same mixing matrix. In fact, in 2.1.4, we have simply assumed the operators to be real, so they were identical to their conjugates. However, in this thesis, we have discussed solely complex operators, so we have to discuss conjugate operators. In particular, since the theories we are looking at are chiral, simple complex conjugation will flip the chirality of the action. Thus, operators will have a different mixing matrix than their complex conjugated counterpart. The easiest way to fix this is to introduce a reshuffling of the operators and define conjugation of operators as complex conjugation plus the reshuffling. To illustrate this point, we are going to look at an example.

Consider the following two operators

$$O_1 = \mathrm{tr}(\phi_1\phi_2\phi_1^\dagger) \quad \text{and} \quad O_2 = \mathrm{tr}(\phi_2\phi_1\phi_1^\dagger). \tag{4.40}$$

These two operators possess eclectic field content and a quick calculation using the dilatation operator shows that

$$D_1 \begin{pmatrix} O_1 \\ O_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} O_1 \\ O_2 \end{pmatrix}, \tag{4.41}$$

while the hermitian conjugates fulfil

$$D_1 \begin{pmatrix} O_1^\dagger \\ O_2^\dagger \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} O_1^\dagger \\ O_2^\dagger \end{pmatrix}. \tag{4.42}$$

Clearly, the two equations are not identical, but it is very simple to put them in an identical form by simply defining

$$\bar{O}_1 = O_2^\dagger \quad \text{and} \quad \bar{O}_2 = O_1^\dagger. \tag{4.43}$$

These operators fulfil the same mixing matrix as the operators O_1 and O_2, and consequently, following the logic from section 2.1.4, the two point function is of the form

$$\begin{pmatrix} \langle \bar{O}_1(x_1)O_1(x_2)\rangle & \langle \bar{O}_1(x_1)O_2(x_2)\rangle \\ \langle \bar{O}_2(x_1)O_1(x_2)\rangle & \langle \bar{O}_2(x_1)O_2(x_2)\rangle \end{pmatrix} = \begin{pmatrix} 0 & \frac{c}{(x_1-x_2)^6} \\ \frac{c}{(x_1-x_2)^6} & \frac{c\ln(\mu^2(x_1-x_2)^2)}{(x_1-x_2)^6} \end{pmatrix}, \tag{4.44}$$

which is easily verified by explicitly calculating the Feynman diagrams. The above argument is easily generalised and justifies our procedure of calculating the dilatation operator.

The example from the last paragraph leads to another interesting question. While we have the mixing matrix up to one-loop in (4.41), there are higher loop contributions. In this specific case, the higher loop contributions change the mixing matrix to

$$\begin{pmatrix} 0 & f(\xi) \\ 0 & 0 \end{pmatrix}. \tag{4.45}$$

Thus, we see here that the Jordan block structure is actually preserved by higher loop-corrections.[4] The question then arises, whether the Jordan block structure is always

[4] We ignore double trace couplings here. Including them might break the Jordan block.

preserved. An answer to the negative was given in [58], taking the following slightly more complicated example.

In [58] the following operators were considered

$$O_1 = \mathrm{tr}(\phi_1^3\phi_2\phi_2^\dagger), \quad O_2 = \mathrm{tr}(\phi_1^2\phi_2\phi_1\phi_2^\dagger), \quad O_3 = \mathrm{tr}(\phi_1\phi_2\phi_1^2\phi_2^\dagger) \quad \text{and} \quad O_3 = \mathrm{tr}(\phi_1^3\phi_2^\dagger\phi_2). \tag{4.46}$$

Drawing all possible Feynman diagrams we see that the mixing matrix for this set of operators has the form

$$\delta D = \begin{pmatrix} 0 & -4\xi^2 + \mathcal{O}(\xi^6) & \mathcal{O}(\xi^4) & \mathcal{O}(\xi^6) \\ 0 & \mathcal{O}(\xi^6) & -4\xi^2 + \mathcal{O}(\xi^8) & \mathcal{O}(\xi^4) \\ 0 & \mathcal{O}(\xi^4) & \mathcal{O}(\xi^6) & -4\xi^2 + \mathcal{O}(\xi^8) \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.47}$$

At one-loop, i.e., at order ξ^2, this mixing matrix is already in Jordan normal form up to normalisation. It is a single Jordan block of size four and generalised eigenvalue zero. However, including higher loop correction will break the Jordan block down to size two and two new eigenstates with eigenvalues $\mathcal{O}(\xi^3)$ will emerge. According to [58] the new eigenstates are

$$O_\mp = \xi^2 O_2 \mp 2i\xi O_3 - 4O_4\,. \tag{4.48}$$

The example from the last paragraph reveals the intricacies of the spectral problem in the strongly-twisted theories. Whether the dilatation operator is diagonalisable or not is arguably the most important aspect of the spectral problem. However, it is not a question that is closed under a one-loop analysis, because arbitrarily small perturbations of a non-diagonalisable matrix often[5] render it diagonalisable. Even if the higher-loop corrections do not lead to a diagonalisable dilatation operator, as in the example above, they might still reduce the size of the Jordan blocks and yield new eigenvalues at higher orders. Furthermore not only to the eigenvalues receive corrections depending on the coupling constant, but the eigenstates also mix with ξ-dependent coefficients. All of these complications need to be disentangled if one desires an all-loop or even fully non-perturbative result.

In this section, we connected the known differences between unitary and logarithmic CFT's with our own results for the strongly-twisted theories. Before concluding, we now want to discuss how our results for the one-loop Bethe equations fit into a possible generalisation to higher loops. The picture we will develop also allows us to discuss some relations to previous work, in particular [54], concerning possible differences and tests to match our results with theirs.

4.6 Higher Loops and Assymptotic Bethe Equations

So far, with the exception of the introductory chapter and a few comments in the last section, our analysis was solely at one-loop level. While the latter is a prerequisite to obtain full results at any value of the coupling constant, it is of course not the full story. In order to proceed, we now start a discussion on the procedure to include higher loops. Twisted, but unscaled all-loop asymptotic Bethe equations exist [47]. The usual procedure to solve these equations is perturbatively around a one-loop solution, highlighting the importance of the one-loop analysis. In the following, we show how to scale the momentum-carrying roots with regards to higher loop corrections.

[5] We will not be precise by what we mean by often. One would have to somehow measure the space of perturbations, which can be done in several different ways. Usually, the perturbation which retain the Jordan block structure form a space with dimension less than the space of all perturbations.

For a higher loop analysis we introduce Zhukovsky variables $x^{\pm}$ [90] as the solutions to the quadratic equation

$$x^{\pm} + \frac{1}{x^{\pm}} = \frac{u \pm i/2}{g}\,, \tag{4.49}$$

where for now we are working with the unscaled parameters. We can expand $x^{\pm}$ in a series of the coupling constant as

$$x^{\pm} = \frac{u \pm i/2}{g} - \frac{g}{u \pm i/2} + \mathcal{O}(g^3)\,. \tag{4.50}$$

From [47] we know that the relation between the momentum of a right-moving excitation and these new variables is

$$e^{ip} = q^2 \frac{x_4^+}{x_4^-}\,. \tag{4.51}$$

As before, we want the lattice momentum to be finite in the double scaling limit. In particular, since we are doing a perturbative analysis, we want it to be finite at every order in ξ. The question then becomes: Can we find a way to scale u_4 and x_4 to guarantee a finite e^{ip}. The natural first attempt is

$$u = \frac{-i}{2} - i\alpha q^{-2}\,, \tag{4.52}$$

where we dropped the subscript on u and allow α to be a power series[6] in ξ^2

$$\alpha = \sum_{j=1}^{\infty} \alpha_j \xi^{2j}. \tag{4.53}$$

Comparison with (4.49) shows that this would be correct, if we could ignore the $1/x^+$ term. To account for this, we just have to add an extra term and end up with

$$u = \frac{-i}{2} - i\alpha q^{-2} + i\xi^2 \alpha^{-1}\,. \tag{4.54}$$

We did not see the extra term in the one-loop analysis since it comes with an extra factor of ξ^2. Using the above scaling for u determines $x^{\pm}$ according to

$$x^+ = \frac{-i\alpha}{\xi q}\,, \quad x^- = \frac{-iq}{\xi}(1 - \xi^2\alpha^{-1}) + \mathcal{O}(q^{-2})\,. \tag{4.55}$$

It follows that order by order in the perturbative expansion e^{ip} is indeed finite as $q \to \infty$

$$e^{ip} = \frac{\alpha}{1 - \xi^2\alpha^{-1}}\,. \tag{4.56}$$

Thus, we indeed achieved finiteness of the lattice momenta. The observables in the CFT are, however, the two-point functions. Therefore, we should also demand a finite $q \to \infty$ limit of the anomalous dimension. The contributions of individual Bethe roots to the anomalous dimension are indeed finite as can be seen from the dispersion relation

$$\gamma_s = 2ig\left(\frac{1}{x^+} - \frac{1}{x^-}\right) = -2\xi^2\alpha^{-1} + \mathcal{O}(q^{-2})\,. \tag{4.57}$$

[6] It might be worth allowing for odd powers of ξ see the discussion in the last section. However, the analysis here will not change.

Since the anomalous dimension is just the sum of these finite parts, it is finite itself as well. Finally putting all of the above together we can get an asymptotic dispersion law

$$\gamma_s = \sqrt{1 - 4\xi^2 e^{-ip}} - 1\,. \tag{4.58}$$

Should one desire to look at left moving excitations instead, one can repeat the above analysis without major changes. Unfortunately, scaling the auxiliary roots is much more complicated, and we were not able to find a general form to yield finite $q \to \infty$ results.

Conceptually, we should also be sceptical of the standard lift to higher loops because as mentioned before some solutions appear only at higher loops. In principle, their appearance could be due to a one-loop solution splitting into two or several at higher loops. However, since solutions even appear in eclectic sectors, where we were not able to find consistent one-loop Bethe equations, this splitting scenario seems unlikely. From the viewpoint of Bethe equations, the emergence of additional solutions remains mysterious.

In [54], the authors already presented asymptotic Bethe equations for the strongly-twisted models. While taking the $q \to \infty$ limit, they kept the coupling constant ξ finite. Only afterwards did they perform the perturbative expansion in ξ. In contrast, we took the perturbative expansion of twisted $\mathcal{N}$=4 SYM first and then took the $q \to \infty$ limit afterwards. The two approaches differ in the order of limits taken, and it is a priori unclear whether they should give the same results. We do, however, find agreement in the anomalous dimensions as described in the following.

Thus, let us also do the scaling analysis of the u_4 roots described above, while keeping ξ fixed. From (4.49) the following relation immediately follows

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{iq}{\xi}\,. \tag{4.59}$$

As a result we can obtain the leading order of q assuming again that the lattice momentum remains finite in the large q limit

$$x^+ \sim q^{-1} \quad \text{and} \quad x^+ \sim q\,. \tag{4.60}$$

Expanding (4.49) in inverse powers of q then lets us solve for $x^\pm$ order by order. We find

$$x^+ = \frac{\xi}{q(u+i/2)} + \mathcal{O}(q^{-2}) \quad \text{and} \quad x^- = \frac{q}{\xi}((u-i/2)) + \mathcal{O}(q^{-2})\,. \tag{4.61}$$

We can then use this expression to find the contributions to the anomalous dimension of the individual Bethe roots

$$\gamma_s = 2ig\left(\frac{1}{x^+} - \frac{1}{x^-}\right) = 2i(u + \frac{i}{2}) + \mathcal{O}(q^{-2})\,. \tag{4.62}$$

In order for this expression to agree with the anomalous dimension (4.57) we require

$$u_4 = \frac{-i}{2} + i\xi^2\alpha\,. \tag{4.63}$$

The ABA eqautions for the of the broken $\mathfrak{su}(2|3)$ sector given in [54] match the Bethe equations for the three-scalar sector (3.68) to (3.70) derived in this thesis. In order to show this, we plug the above version of u_4 into the ABA equations and restrict the magnon-numbers to the three-scalar sector and expand up to order ξ^2. The agreement between the anomalous dimension suggests that the order of limits only leads to superficial differences and is irrelevant for physical quantities.

An important note is that in [54] the authors claim to find the Bethe equations for the ϕ_1, ϕ_2, ϕ_3 sector which is eclectic. In fact, trying to do a one-loop Bethe ansatz in this case is bound to fail as explained in 3.3.2. However, this can be fixed by using a different convention for the scaling of q. Letting some of the $q_i \to 0$, while keeping $\xi = g/q_i$ fixed also produces a strongly-twisted theory. The twisted equations used in the appendix of [54] are identical to the ones used in their introduction and ours after the replacement $q_1 \to q_1^{-1}$ and $q_2 \to q_2^{-1}$. This replacement leads to an identical theory with the fields being redefined. In particular the $\{\phi_1, \phi_2, \phi_3\}$ sector becomes the $\{\phi_1^\dagger, \phi_2^\dagger, \phi_3\}$, which is equivalent to our three-scalar sector. Thus, we should find the same results for physical quantities.

Chapter 5

Conclusions and Outlook

The main topic of this thesis is a systematic investigation of the spectral problem of strongly-twisted $\mathcal{N} = 4$ SYM. The initial aim was to adapt the solution of the spectral problem of fully-fledged $\mathcal{N} = 4$ SYM to its doubly-scaled cousins. With this in mind, we give a detailed account of the one-loop part of the problem. By investigating these models, we may hope to gain additional insights into the role and origins of integrability of quantum field theories, since the Feynman diagrammatics of said models is significantly simpler. On the other hand, we also show that intricacies arise when scaling twisted $\mathcal{N} = 4$ SYM, questioning the validity and obscuring the interpretation of many of the results provided by integrability. In particular, the non-unitarity of the models under investigation leads to complications, rendering the very definition of the spectral problem - diagonalisation of the dilatation operator - ill-defined.

After a quick introduction into the relevant literature, we start by scaling the dilatation operator of the unscaled γ-twisted model. Due to the limited number of free parameters, whose scaling behaviour is known explicitly, this procedure is guaranteed to give the correct dilatation operator of the strongly-twisted models. We show, with the help of projection operators, how to write the one-loop dilatation operators of the scaled theories as a slight alteration of the one of the untwisted theory. Equipped with these objects of study, we then discuss several Bethe ansätze to diagonalise the corresponding spin-chain Hamiltonians in different sectors. We are able to show several simplifications to carry over from the level of the Feynman diagrammatics to the application of the Bethe ansatz. Most notably, the nesting procedure becomes obsolete in the sector consisting of two different scalars and one type of derivative. Also, several computations are simpler than in unscaled $\mathcal{N} = 4$ SYM. However, many sectors become unaccessible to the Bethe ansatz, which leads us to define the notion of eclectic spin-chains. For eclectic spin-chains, the Hamiltonian is nilpotent. We present an intuitive argument for this fact and are able to prove it. Moving to the largest non-eclectic sectors, instead of attempting a Bethe ansatz, which might turn out to be inapplicable, we introduce a prescription to scale the Beisert-Roiban equations [47]. There are several arguments for a unique correct scaling of the momentum-carrying roots. In contrast, for the auxiliary roots, the scaling procedure we suggest is ad hoc and mostly motivated by simplicity and inspection of the resulting equations. Thus, the derived Bethe equations in the maximal non-eclectic sectors, are only conjectural in nature, however, well-motivated. Assuming the scaling procedure to be correct, these equations together with the observation of nilpotency in eclectic sectors together, give the complete solution of the one-loop spectral problem.

In order to motivate the derived Bethe equations and gain further insights into the structure of the one-loop dilatation operator, we analyse said equations and their solutions

in chapter four. Once more, we can observe that the Bethe equations of the strongly-twisted models are considerably simpler than those of their unscaled cousins. For the two-scalar sector, we derive a bounded region in the complex γ-plane in which the anomalous dimensions lie for a given magnon number M. Explicit bounds are given for $M \leq 4$. Furthermore, we show that in this sector the Hamiltonian is diagonalisable and all energies are found by the Bethe ansatz. The Bethe equations of the three-scalar sector are also simpler than those of the unscaled model. While we are not able to systematically obtain all solutions to these equations for $M \geq 3$, we display several explicit solutions. Notably, in this sector, the Hamiltonian is no longer diagonalisable, and Jordan blocks of size two appear. For $M = 2$ and $M = 3$ we give the explicit form of the states forming the Jordan blocks. Using these as inspiration, we conjecture a relationship between a symmetric distribution of Bethe roots in the complex plane and the appearance of Jordan blocks. So far, there is one symmetric distribution of Bethe roots for all Jordan blocks found in this sector.

For the sectors with larger field content results remain more scarce. Interestingly, the Bethe equations of these larger sectors decouple into two groups. The first group of equations is given by the two- or three-scalar equations, depending on the number of flavours involved, appended by additional factors. These can be determined by solving the second group of equations, which is independent of the first. Furthermore, the second group bears striking similarity with the standard Bethe equations of, for example, a $\mathfrak{su}(2,2)$ spin-chain. In these larger sectors, additional Jordan blocks of arbitrary size appear. Possibly, all of the Jordan blocks of size larger than two can be explained by the appearance of certain wall structures we describe. We present the explicit Jordan normal form of the one-loop dilatation operator in the maximal non-eclectic sectors for small classical dimensions in sections 4.3 and 4.4. Remarkably, all observed Jordan blocks have generalised eigenvalue zero. It is unclear whether this is general or other generalised eigenvalues appear for larger spin-chain length.

We end with a few peripheral observations. Firstly, we suggest a way to move to higher loops. This leads us, secondly, to a comparison of our work to asymptotic Bethe equations derived in [54]. If these equations are expanded up to one-loop order, they agree with the ones we derived. Thirdly, we derive the form of two-point correlation functions in the strongly-twisted theories, and in particular, how it is affected by the Jordan blocks of the dilatation operator. Finally, we also discuss a connection between Jordan blocks and perturbation theory. The Jordan block structure is sensitive to arbitrary small perturbations, and consequently, great care has to be taken when using approximations like the expansion in a coupling constant. Thus, higher loops can break the Jordan blocks. We also display how this can lead to a mixing of loop orders following the example from [58] closely.

All in all, the one-loop spectral problem of strongly-twisted $\mathcal{N} = 4$ SYM is significantly more intricate than one might naively have expected. The lack of solutions to the Bethe equations due to the Jordan blocks raises doubt whether the success story of solving the spectral problem of $\mathcal{N} = 4$ SYM is easily repeatable for its strongly-twisted cousins. Certainly, new ideas are needed, if one wishes to rigorously justify adaptations of the Bethe ansatz and the quantum spectral curve to these models. On the other hand, this might also represent a unique opportunity to investigate integrability and in particular the Bethe ansatz for non-unitary models and non-hermitian operators, respectively. After all, in some sense, these models do appear simpler, not only on the Feynman diagrammatic level but also at the level of the Bethe equations.

Regarding future research, there are numerous different avenues to explore. As the

focus of this thesis already exemplifies, it is critical to obtain a better understanding of the Jordan blocks appearing as constituents of the dilatation operator. In particular, it would be interesting to classify the Jordan blocks according to their origin. We only started doing that, establishing two categories of Jordan blocks - those created by walls and what we called accidental Jordan blocks. However, it has remained unclear whether these are actually all possible types. The concrete origin of the accidental Jordan blocks also presents a pressing open question. An equally important problem is to find the exact explicit form of the Jordan blocks. In particular, the size of the Jordan blocks and their generalised eigenvalues need to be determined. The Bethe ansatz itself is used to find eigenvalues and eigenstates, and as such it does not yield all the states composing a Jordan block. In fact, it does not even always find the corresponding eigenstate. The question arises, whether the Bethe ansatz can be adapted such that it can find the whole Jordan blocks. This has been done for a specific model in [91], but so far no *systematic* Bethe ansatz exists for Jordan blocks.

A second and possibly the most obvious avenue for further research lies in the higher-loop and finite coupling spectral problem. Historically, the one-loop solution was paramount as a first stepping stone to the finite coupling solution of the spectral problem of the untwisted $\mathcal{N} = 4$ SYM. We hope that a similar approach is fruitful in the strongly-twisted models. Many higher loop results have already been worked out. However, these results were obtained by twisting and scaling the asymptotic Bethe ansatz and the quantum spectral curve of untwisted $\mathcal{N} = 4$ SYM. Thus, these results are mostly limited to the diagonalisable part of the dilatation operator. The bottom-up approach of starting with the one-loop problem and systematically going to higher loops is both a good check on the applicability of the scaled quantum spectral curve as well as a promising strategy to describe the Jordan blocks. In order to take this approach, most likely a better understanding of the underlying symmetry groups and in particular their representations will be needed. While the groups are given in this thesis, finding the representations and their relation to the Bethe equations remains a task for the future.

The spin-chains discussed in this thesis are motivated by their relation to quantum field theory. However, they also represent interesting physical models themselves. Outside the quantum field theory picture, one can extend these spin-chains by adding additional types of excitations. For example, another type of excitation moving right past all other excitations could be included. Of course, many other extensions can be imagined as well. Investigating these extensions might also lead to a better understanding of the Jordan blocks. Since they are absent in the two-scalar sector and appear in the three-scalar sector, we might ask what about a 'four-scalar' sector? Do Jordan blocks appear, and if so, why?

Finally, the appearance of logarithmic CFTs as limiting points of ordinary CFTs is not new. For two-dimensional theories, the paper [92] gives several examples. However, as far as we know, the double-scaling procedure is the first example of this phenomenon in four-dimensional theories. The exact relation between the double-scaling procedure and the known examples of the appearance of logarithmic CFTs is undoubtedly worth exploring. Furthermore, it is an interesting open question whether results can be transferred between the known examples and the strongly-twisted theories.

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Appendix A

Twisted One-Loop Bethe Equations

In this appendix, we write down the twisted Bethe equations from [47] that we need. The Dynkin diagram of $\mathfrak{su}(2,2|4)$ admits various gradings, which leads to different sets of Bethe equations. Here we will use the "Beauty" grading [17] and the ABA grading [20].

A.1 "Beauty" Grading

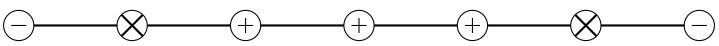

Figure A.1: The Dynkin diagram of $\mathfrak{su}(2,2|4)$ in the Beauty grading.

	4	5	6	7
4	ϕ_2	ϕ_3	ψ_{41}	ψ_{42}
3	$\phi_3^\dagger$	$\phi_2^\dagger$	ψ_{11}	ψ_{12}
2	$\bar{\psi}_{3\dot{1}}$	$\bar{\psi}_{2\dot{1}}$	$\partial_{1\dot{1}}$	$\partial_{2\dot{1}}$
1	$\bar{\psi}_{3\dot{2}}$	$\bar{\psi}_{2\dot{2}}$	$\partial_{1\dot{2}}$	$\partial_{2\dot{2}}$

Table A.1: Single excitations of the full $\mathcal{N}=4$ SYM spin chain in the "Beauty" grading. The table should be read as follows: Consider an state with the non-zero Ks being $K_j = K_{j+1} = \cdots = K_k = 1$, where $1 \leq j \leq 4 \leq k \leq 7$. The corresponding excitation, over a vacuum of ϕ_1s, is the one listed at row j and column k.

For this grading we only consider the β-twist. We thus set $\gamma_1 = \gamma_2 = \gamma_3 = \beta$ and $q = e^{-i\beta/2}$. The twisted Bethe equations are [47]:

$$1 = q^{2K_4-4K_5+2K_6} \prod_{j=1}^{K_4} \frac{u_{4,j}+i/2}{u_{4,j}-i/2}\,, \tag{A.1}$$

$$1 = \prod_{j\neq k}^{K_1} \frac{u_{1,k}-u_{1,j}-i}{u_{1,k}-u_{1,j}+i} \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+i/2}{u_{1,k}-u_{2,j}-i/2}\,, \tag{A.2}$$

$$1 = \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+i/2}{u_{2,k}-u_{1,j}-i/2} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}-i/2}{u_{2,k}-u_{3,j}+i/2}\,, \tag{A.3}$$

$$\begin{aligned} 1 = {} & q^{2K_4-4K_5+2K_6} \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}-i/2}{u_{3,k}-u_{2,j}+i/2} \\ & \times \prod_{j\neq k}^{K_3} \frac{u_{3,k}-u_{3,j}+i}{u_{3,k}-u_{3,j}-i} \prod_{j=1}^{K_4} \frac{u_{3,k}-u_{4,j}-i/2}{u_{3,k}-u_{4,j}+i/2}\,, \end{aligned} \tag{A.4}$$

$$\begin{aligned} \left(\frac{u_{4,k}+i/2}{u_{4,k}-i/2}\right)^L = {} & q^{-2L-2K_3+6K_5-4K_6} \prod_{j=1}^{K_3} \frac{u_{4,k}-u_{3,j}-i/2}{u_{4,k}-u_{3,j}+i/2} \\ & \times \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j=1}^{K_5} \frac{u_{4,k}-u_{5,j}-i/2}{u_{4,k}-u_{5,j}+i/2}\,, \end{aligned} \tag{A.5}$$

$$\begin{aligned} 1 = {} & q^{4L+4K_3-6K_4+2K_6} \prod_{j=1}^{K_4} \frac{u_{5,k}-u_{4,j}-i/2}{u_{5,k}-u_{4,j}+i/2} \\ & \times \prod_{j\neq k}^{K_5} \frac{u_{5,k}-u_{5,j}+i}{u_{5,k}-u_{5,j}-i} \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}-i/2}{u_{5,k}-u_{6,j}+i/2}\,, \end{aligned} \tag{A.6}$$

$$\begin{aligned} 1 = {} & q^{-2L-2K_3+4K_4-2K_5} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}-i/2}{u_{6,k}-u_{5,j}+i/2} \\ & \times \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}+i/2}{u_{6,k}-u_{7,j}-i/2}\,, \end{aligned} \tag{A.7}$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k}-u_{6,j}+i/2}{u_{7,k}-u_{6,j}-i/2} \prod_{j\neq k}^{K_7} \frac{u_{7,k}-u_{7,j}-i}{u_{7,k}-u_{7,j}+i}\,. \tag{A.8}$$

The elementary excitations are listed in Table A.1. The momentum constraint (A.1) agrees with Eq. (3.92), where $K_R = K_4 - K_5$ and $K_L = K_5 - K_6$.

A.2 ABA Grading

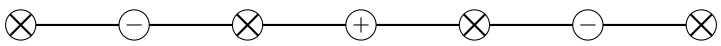

Figure A.2: The Dynkin diagram of $\mathfrak{su}(2,2|4)$ in the ABA grading.

	4	5	6	7
4	ϕ_2	ψ_{41}	ψ_{42}	ϕ_3
3	$\bar{\psi}_{3\dot{1}}$	$\partial_{1\dot{1}}$	$\partial_{2\dot{1}}$	$\bar{\psi}_{2\dot{1}}$
2	$\bar{\psi}_{3\dot{2}}$	$\partial_{1\dot{2}}$	$\partial_{2\dot{2}}$	$\bar{\psi}_{2\dot{2}}$
1	$\phi_3^\dagger$	ψ_{11}	ψ_{12}	$\phi_2^\dagger$

	4	5	6	7
4	$\phi_3^\dagger$	ψ_{11}	ψ_{12}	$\phi_2^\dagger$
3	$\bar{\psi}_{3\dot{1}}$	$\partial_{1\dot{1}}$	$\partial_{2\dot{1}}$	$\bar{\psi}_{2\dot{1}}$
2	$\bar{\psi}_{3\dot{2}}$	$\partial_{1\dot{2}}$	$\partial_{2\dot{2}}$	$\bar{\psi}_{2\dot{2}}$
1	ϕ_2	ψ_{41}	ψ_{42}	ϕ_3

Table A.2: Single excitations of the full $\mathcal{N}=4$ SYM spin chain in the ABA grading. The left table is in the conventions of [47], while the right is the R-symmetry rotated variant we use in connection with the β-twist. The notation is the same as Table A.1.

	γ_3-twist	β-twist non-rotated	β-twist rotated
t_0	$-K_1-K_3+2K_4-K_5-K_7$	$2K_4-2K_5$	$2K_4-2K_5-2K_7$
t_1	$L+K_3-2K_4+K_5$	0	$-2K_4+2K_5+2K_7$
t_3	$L-K_1-K_7$	$2K_4-2K_5$	0
t_4	$-2L+2K_1+2K_7$	$-2L+2K_3-2K_5$	$-2L+2K_1+2K_7$
t_5	$L-K_1-K_7$	$2L-2K_4+2K_3$	$2L-2K_1-2K_7$
t_7	$L+K_3-2K_4+K_5$	$2(L-2K_4+K_3+K_5)$	$2L-2K_1-2K_4+2K_5$

Table A.3: Twist factors for the ABA Bethe equation.

For the Bethe equations in the ABA grading we will consider two different twists. For the γ_3-twist we set $\gamma_1=\gamma_2=0$ and $q=e^{-i\gamma_3/2}$. For the β-twist we set $\gamma_1=\gamma_2=\gamma_3=\beta$ and $q=e^{-i\beta/2}$. The corresponding values of $t_0,\ldots,t_7$ are given in Table A.3. The elementary excitations are listed in Table A.2. Note that for the β-twist we have a version of the equations, where we performed an R-symmetry rotation compared to the conventions of [47].

In the ABA grading the Bethe equations take the following form [47]:

$$1 = q^{t_0} \prod_{j=1}^{K_4} \frac{u_{4,j} + i/2}{u_{4,j} - i/2}\,, \tag{A.9}$$

$$1 = q^{t_1} \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + i/2}{u_{1,k} - u_{2,j} - i/2}\,, \tag{A.10}$$

$$1 = \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + i/2}{u_{2,k} - u_{1,j} - i/2} \times \prod_{j\neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_3} \frac{u_{2,k} - u_{3,j} + i/2}{u_{2,k} - u_{3,j} - i/2}\,, \tag{A.11}$$

$$1 = q^{t_3} \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + i/2}{u_{3,k} - u_{2,j} - i/2} \prod_{j=1}^{K_4} \frac{u_{3,k} - u_{4,j} - i/2}{u_{3,k} - u_{4,j} + i/2}\,, \tag{A.12}$$

$$\left(\frac{u_{4,k} + i/2}{u_{4,k} - i/2}\right)^L = q^{t_4} \prod_{j=1}^{K_3} \frac{u_{4,k} - u_{3,j} - i/2}{u_{4,k} - u_{3,j} + i/2} \times \prod_{j\neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \prod_{j=1}^{K_5} \frac{u_{4,k} - u_{5,j} - i/2}{u_{4,k} - u_{5,j} + i/2}\,, \tag{A.13}$$

$$1 = q^{t_5} \prod_{j=1}^{K_4} \frac{u_{5,k} - u_{4,j} - i/2}{u_{5,k} - u_{4,j} + i/2} \prod_{j=1}^{K_6} \frac{u_{5,k} - u_{6,j} + i/2}{u_{5,k} - u_{6,j} - i/2}\,, \tag{A.14}$$

$$1 = \prod_{j=1}^{K_5} \frac{u_{6,k} - u_{5,j} + i/2}{u_{6,k} - u_{5,j} - i/2} \times \prod_{j\neq k}^{K_6} \frac{u_{6,k} - u_{6,j} - i}{u_{6,k} - u_{6,j} + i} \prod_{j=1}^{K_7} \frac{u_{6,k} - u_{7,j} + i/2}{u_{6,k} - u_{7,j} - i/2}\,, \tag{A.15}$$

$$1 = q^{t_7} \prod_{j=1}^{K_6} \frac{u_{7,k} - u_{6,j} + i/2}{u_{7,k} - u_{6,j} - i/2}\,. \tag{A.16}$$

Appendix B

Nilpotency Proof

In this appendix, we will prove that the Hamiltonian in the eclectic sectors is nilpotent. The final version of this proof was developed by my collaborator Asger Ipsen, based on an earlier less elegant version by myself. It can be found in [1], but this version collects several remarks to render the prove less technical. The theorem can be formulated as follows: For all operators $\mathrm{tr}(A_1 \cdots A_L)$ and all flavours $b \in F$, if there exists A_i such that

$$a_i = \mathcal{F}(A_i) \notin \{b, b_+, \bar{b}_-\} \tag{B.1}$$

then there exists $N > 0$ such that $H^N |A_1 \cdots A_L\rangle = 0$. The theorem holds for the Hamiltonian of the strongly-twisted γ-deformation including the two special cases of the β-deformation and the fishnet theory. The condition of the theorem is a concise version of our previous definition of eclectic field content. More visually we say an operator is not eclectic if it contains fields of at most two flavours and the third conjugate flavour.

As a first step, we realise that under the action of the Hamiltonian density pairs of fields change from chiral into antichiral order, whenever the Hamiltonian density is not zero. The general strategy is then to find a function d from a flavour sequence $(a_1 \cdots a_L)$ to the integers, which is bounded from above, and such that whenever $a_1 a_2$ are in chiral order, we have

$$d(a_1 \cdots a_L) < d(a_2 a_1 \cdots a_L)\,. \tag{B.2}$$

If we find such a function, d will increase whenever the action of H is not zero, but because it is bounded from above, it can not increase indefinitely, hence after a maximum number of applications of H the spin-chain state will be annihilated. There are many different ways to define such a function leading to different proofs. Here, we will only show the most elegant we have found so far.

We define a subsequence $x_1 \cdots x_n$ in a sequence $a_1 \cdots a_L$ as being obtained by deleting some of the a_i and possibly permuting, i.e., there exists increasing indices $\{i_m\}_{m=1,\cdots,n}$ such that

$$(x_1 x_2 \cdots x_n) = (a_{i_1} \cdots a_{i_n}) \tag{B.3}$$

modulo a cyclic shift. The number of times a subsequence $(x_1 \cdots x_n)$ occurs in a sequence $(a_1 \cdots a_L)$ we call mutiplicity and denote it as $\mathrm{mul}[(\mathrm{x_1} \cdots \mathrm{x_n})|(\mathrm{a_1} \cdots \mathrm{a_L})]$. If we denote by P_- and P_+ the sets of chirally ordered and antichirally order pairs respectively we can define the two sets

$$C_\pm = \{(x_1 \cdots x_n) | n \geq 3 \wedge \forall i (x_i x_{i+1}) \notin P_\mp\}, \tag{B.4}$$

These sets can be taken as a generalisation of chiral and antichiral order for subsequences. The idea is now that the Hamiltonian will put chirally ordered subsequences into antichiral

order and hence the number

$$d(a_1 \cdots a_L) = \sum_{(x_1 \cdots x_n) \in C_+} \mathrm{mul}[(\mathrm{x_1} \cdots \mathrm{x_n})|(\mathrm{a_1} \cdots \mathrm{a_L})] - \sum_{(\mathrm{x_1} \cdots \mathrm{x_n}) \in \mathrm{C_-}} \mathrm{mul}[(\mathrm{x_1} \cdots \mathrm{x_n})|(\mathrm{a_1} \cdots \mathrm{a_L})] \tag{B.5}$$

can never decrease. Without loss of generality, we will assume for the rest of the proof that the Hamiltonian acts on the pair $a_1 a_2$, which we implicitly assume to be in chiral order. The only way the multiplicity of a subsequence in a sequence can change, when acting with the Hamiltonian, is if there is an i such that $x_i = a_1$ and $x_{i+1} = a_2$. However, if this is the case, the multiplicity of chirally ordered subsequences can only decrease, while that of antichirally ordered sequences can only increase. We are led to the conclusion that

$$d(a_1 \cdots a_L) \leq d(a_2 a_1 \cdots a_L). \tag{B.6}$$

What is left to show is that for eclectic field content, the equality is not possible. In order to do so, we have to distinguish the following cases.

First assume $a_1 = a$ and $a_2 = a_-$. For eclectic field content it follows that

$$\exists i \text{ such that } a_i \in \{\bar{a}, \bar{a}_-, \emptyset, a_+\}. \tag{B.7}$$

Than

$$(a_2 a_1 a_i) = (a_- a a_i) \in C_+ \,. \tag{B.8}$$

It follows immediately that C_+ and hence d increases.

Secondly assume $a_1 = a$ and $a_2 = \bar{a}_+$. For eclectic field content we either have

$$\exists i \text{ such that } a_i \in \{a_+, \bar{a}, \emptyset\} \tag{B.9}$$

in which case

$$(a_2 a_1 a_i) \in C_+ \tag{B.10}$$

or

$$\exists i < j \text{ such that } a_i = \bar{a}_- \text{ and } a_j = a_- \tag{B.11}$$

in which case

$$(a_2 a_1 a_i a_j) \in C_+ \tag{B.12}$$

or

$$\exists i < j \text{ such that } a_i = a_- \text{ and } a_j = \bar{a}_- \tag{B.13}$$

in which case

$$(a_1 a_2 a_i a_j) \in C_- \,. \tag{B.14}$$

This shows that whenever a_1 and a_2 are in chiral order d increases, when acting with H. Clearly, d is bounded above, since the number of times subsequences can occur in a sequence of given length is large but finite. It follows that repeated action with H will eventually lead to a spin-chain state, without any pairs in chiral order and hence one additional application of H will annihilate the spin-chain state. We should stress that this proof is not well suited to estimate the degree of nilpotency since d usually does not reach values close to its bound and d tends to increase in steps larger than one, but difficult to estimate.

Appendix C

Walls in non-Eclectic Sectors

We have argued that the spectrum of H in certain sectors that we called eclectic, is trivial in the sense that the (generalised) energies of states in this sectors are zero or equivalently $H^n = 0$ for some finite natural number n. The physical reason responsible for the nilpotency is that while the excitations travel around the spin-chain in one direction, there are excitations of different flavour presenting walls for these excitations. However, by careful examination of the dilatation operator, we can also find walls in sectors without eclectic field content. Consequently, also in non-eclectic sectors, certain states will have (generalised) energy zero and Jordan blocks will form. In this appendix, we want to elaborate on this phenomenon on the example of the two-scalar sector with derivatives of the fishnet theory.

Consider the sector defined by the spin-chain sites $\{\partial^k\phi_1, \partial^k\phi_2\}$. The critical realisation that leads us to the walls in this sector is that chirally ordered pairs of neighbouring spin-chain sites present impenetrable obstacles for other excitations. Explicitly, we mean that if we have a chirally ordered pair

$$a_i a_{i+1} = (\partial^k\phi_2)(\partial^l\phi_1) \tag{C.1}$$

at position i and $i+1$ in the spin chain, than $a_{i-1}a_i$ and $a_{i+1}a_{i+2}$ are never in chiral order. Usually, this does not lead to a wall, because the Hamiltonian density can act on the chirally order pair itself, putting it into antichiral order. Put loosely, walls travelling around a spin chain in the same direction as the excitations are not necessarily resulting in annihilated states. However, it immediately follows that, if we can create a stable pair, i.e., one on which the Hamiltonian density is zero, the wall stays in place, and the Hamiltonian on the corresponding subspace is nilpotent. Indeed, since the action of the Hamiltonian density is symmetric under an interchange of k and l any antisymmetric combination will be annihilated by the Hamiltonian, the simplest of which is

$$a_i a_{i+1} = (\partial\phi_2)(\phi_1) - (\phi_2)(\partial\phi_1)\,. \tag{C.2}$$

The above analysis is valid for spin-chains without the zero-momentum constraint imposed. However, given a state with a wall in it, we can just build the symmetric superposition of it and all states related by a cyclic shift. The resulting superposition is invariant under cyclic shifts by definition. We can write this in formulas as follows. Given a state of the spin-chain with a wall, $|W\rangle$, the state

$$|W_0\rangle = \frac{1}{L}\sum_{n=0}^{L-1} U^n\,|W\rangle \tag{C.3}$$

is invariant under cyclic shifts and $H^n |W_0\rangle = 0$ for some finite n. Here, U is the translation operator, shifting all sites by one to the right.[1]

There are two points which are important to stress here but are not yet completely understood. Firstly, while in principle walls can create either Jordan blocks or just isolated states that are annihilated by the Hamiltonian, they will tend to create Jordan blocks of arbitrary sizes as long as the length of the spin chain is large enough and there is more than one excitation. The Jordan blocks appearing in the dilatation operator of the strongly-twisted theories are certainly one of the most fascinating features of the models. However, it is safe to say they are also among the least understood features. They significantly complicate the spectral problem from a conceptual point of view, because they indicate a different structure in the two-point functions; see also section 2.1.4. Secondly, the walls described above are, a priori, only a one-loop feature. They depend explicitly on the local action of the Hamiltonian. If the Hamiltonian was not of nearest neighbour type, the excitations might be able to jump over the wall. Of course, it is a possibility that larger structures like the above exist, which span over several spin-chain sites and are annihilated by the higher-loop Hamiltonian. However, in contrast to the example from [58], it is unclear for this model why the action of the higher-loop Hamiltonian should conspire to allow such structures.[2]

[1] We can just as well define U to shift everything to the left, of course.

[2] For the example from [58], one can argue by the available fishnet diagrams that the Jordan block should be protected. In contrast, here it is the explicit coefficients that have to cancel, rendering a diagrammatic explanation along the same line infeasible.

Appendix D

Energies of States With Low Classical Dimension

In this appendix, we give the energies of some additional states. As mentioned in the main text, JB(E, d) stands for a Jordan block of size d with generalised energy E. In table D.1 we see the energies of states in the fishnet model with $\Delta_0 = 6$. We assumed that the number of ϕ_1 is at least as large as the number of ϕ_2, resulting in twice as many states of a given energy in the case of equality. Descendants are also shown, they have the same energy as some lower lying states. States corresponding to protected operators like $\mathrm{tr}(\phi_1^6)$ are not shown. Some results are only numerical and are rounded to two digits after the decimal point.

The same analysis is done for the maximal non-eclectic sectors of the strongly-β-twisted models. The energies of states with $\Delta_0 = 4.5$ are shown in table D.2. Again, while descendants are included, protected operators are excluded. Numerical results are rounded, once again, to two digits after the decimal point. The number of $c_3^\dagger$ is set to zero to impose non-eclectic field content. Other maximal non-eclectic sectors exist, however, the dilatation operator acts on them in the same way.

ϕ_1	ϕ_2	∂	E	multiplicity
2	1	3	-2	20
			-2/3	36
			0	216
			1/2	16
			1	40
3	1	2	-2	10
			-1	6
			0	56
			1	1
			$0.5 + 1.32i$	16
			$0.5 - 1.32i$	16
			-1.18	9
			$0.76 + 0.74i$	9
			$0.76 - 0.74i$	9
2	2	2	0	62
			JB(0,3)	60
			$2i$	1
			$-2i$	1
			$2\sqrt{2}$	14
			$-2\sqrt{2}$	14
			-1.15	13
			1.15	13
4	1	1	-2	4
			0	4
			1.48	4
			$-0.24 + 1.63i$	4
			$-0.24 - 1.63i$	4
3	2	1	JB(0,4)	4
			-3.24	4
			1.24	4
			$-0.62 + 0.62i$	4
			$-0.62 - 0.62i$	4
			$1.62 + 1.62i$	4
			$1.62 - 1.62i$	4
5	1	0	-2	1
4	2	0	$-2\sqrt{3}$	1
			0	1
			$2\sqrt{3}$	1
3	3	0	-4	1
			0	1
			$2 + 2\sqrt{3}i$	1
			$2 - 2\sqrt{3}i$	1

Table D.1: Energies of states in the fishnet model with $\Delta_0 = 6$.

$a_1^\dagger$	$a_2^\dagger$	$b_1^\dagger$	$b_2^\dagger$	$c_1^\dagger$	$c_2^\dagger$	$c_3^\dagger$	$c_4^\dagger$	E
1	0	0	0	3	1	0	3	$\{-2, 0, 1/2 \pm \sqrt{7}i/2\}$
				2	2	0	3	$\{0, \pm 2\sqrt{2}, \mathrm{JB}(0,3)\}$
				4	1	0	2	$\{-2, 1 \pm \sqrt{3}i\}$
				3	2	0	2	$\{-1 \pm \sqrt{5}, 2, -0.47, 0.24 \pm 2.04i, \mathrm{JB}(0,3)\}$
				4	2	0	1	$0, \pm 2\sqrt{2}$
				3	3	0	1	$\{-3.17, 1.59 \pm 2.75i, \mathrm{JB}(0,3)\}$
				4	3	0	0	-2
0	0	1	0	4	1	0	4	-2
				3	2	0	4	$\{-1 \pm \sqrt{5}, 2\}$
				4	2	0	3	$0, \pm 2\sqrt{2}$
				3	3	0	3	$\{-3.17, 0, 1.59 \pm 2.75i, \mathrm{JB}(0,2)\}$
				4	3	0	2	$\{-2, 1 \pm \sqrt{3}i\}$
3	0	0	0	2	1	0	0	0
2	1	0	0	2	1	0	0	$\{0, 0, 1\}$
2	0	1	0	2	1	0	2	$\{-2, -2/3, 0^{\otimes 5}, 1, 1\}$
				3	1	0	1	$\{-2, -1 \pm i, 2, 2, 1/2 \pm \sqrt{7}i/2\}$
				2	2	0	1	$\{\pm 2\sqrt{2}, \mathrm{JB}(0,2)^{\otimes 6}\}$
				3	2	0	0	$\{-2, 1/2 \pm \sqrt{7}i/2\}$
1	1	1	0	2	1	0	2	$\{-2, -2, -2/3, 0^{\otimes 8}, 1^{\otimes 4}\}$
				3	1	0	1	$\{-2^{\otimes 3}, -1 \pm i, 2^{\otimes 3}, 1/2 \pm \sqrt{7}i/2, 1/2 \pm \sqrt{7}i/2\}$
				2	2	0	1	$\{(\pm 2\sqrt{2})^{\otimes 3}, \mathrm{JB}(0,2)^{\otimes 9}\}$
				3	2	0	0	$\{-2, -2, 2, 1/2 \pm \sqrt{7}i/2\}$
1	0	2	0	3	1	0	3	$\{-2, 1/2 \pm \sqrt{7}i/2\}$
				2	2	0	3	$\{\pm 2\sqrt{2}, \mathrm{JB}(0,2), \mathrm{JB}(0,2)\}$
				3	2	0	2	$\{-2, -1 \pm i, 2, 2, 1/2 \pm \sqrt{7}i/2\}$
1	0	1	1	3	1	0	3	$\{-2, -2, 2, 1/2 \pm \sqrt{7}i/2\}$
				2	2	0	3	$\{\pm 2\sqrt{2}, \pm 2\sqrt{2}, \mathrm{JB}(0,2)^{\otimes 3}\}$
				3	2	0	2	$\{-2^{\otimes 3}, 2^{\otimes 3}, -1 \pm i, 1/2 \pm \sqrt{7}i/2, 1/2 \pm \sqrt{7}i/2\}$

Table D.2: Energies of states with $\Delta_0 = 4.5$ in the strongly-β-twisted theory. Every energy or Jordan block that is repeated three times or more is indicated by $E^{\otimes \mathrm{multiplicity}}$. Again states, which are obtained by choosing an equivalent set of spinor indices or interchaning the number of $c_1^\dagger$ and $c_2^\dagger$ are not shown.

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